# Network formation, cost-sharing and anti-coordination *

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**Abstract**

This paper presents a non-cooperative model of network formation where agents link to play anti-coordination games. Links are costly but, unlike in standard one-sided models, the cost is shared between the two players involved in a link. We show that the set of Nash equilibria of the resulting social game shrinks as the shares of the link cost are more equal. In the extreme case in which each agent pays half of the cost, there is a unique equilibrium. We also show that, as usual in the literature of network formation, there is a general misalignment between the stable and efficient states of the game.

**Keywords**: anti-coordination, Nash equilibrium, efficiency, one-sided models, two-sided models.

**JEL Classification Numbers**: A14, C72, C78

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# 1 Introduction

Networks have been increasingly studied in the last years since they are crucial in determining the nature of many social and economic outcomes (see e.g., Ellison, 1993; Young, 1993; Anderlini and Ianni, 1996; Goyal, 1996; Morris, 2000). Recently, several authors have studied how networks emerge and how the decisions of individuals contribute to the network formation (see e.g., Aumann and Myerson, 1989; Jackson and Wolinsky, 1996; Bala and Goyal, 2000). Two major models of network formation have been proposed: one-sided and two-sided. Links is costly. In the former case, agents unilaterally propose to form links and pay the full cost of them. Consequently, the network formation process can be formulated using a non-cooperative approach and thus the standard Nash equilibrium concept applies. In the latter case, links are formed bilaterally since the cost of a link is divided among the two agents involved in it. Here, the notion of stable networks rests on pairwise incentive compatibility, thus making this approach closer to cooperative game theory. In the present paper, we develop a non-cooperative model of network formation (thus, it has the advantages of a one-sided model) but nevertheless it allows the implement of more realistic ways of dividing the cost of the link. Normally, the two agents involved incur in some cost. Most frequently the cost is not equal since the agent proposing or initiating the link contributes more. An illustrative example of this "cost-sharing" model is the link established between two scientists when writing a paper; if the person initiating the link is interpreted as the one who writes the first version of the manuscript, it is reasonable to assume that she is exerting a higher effort. Another stylized example is found in mobile telephone communication networks in USA, where incoming calls are costly. More precisely, a person receiving a phone call is charged an amount representing a small percentage of the cost, that will naturally be paid in the larger proportion by who makes the call (the agent initiating the link). Specifically, we present a model in which we assume that (active) agents can unilaterally propose links to other (passive) agents. The minimum cost required for the link to form is $c > 0$. The proposer or *active agent* of the link incurs in a sunk cost of $\lambda c$ where $\lambda \in [1/2, 1]$, whereas the cost incurred by the proposed or *passive agent* in case of accepting the offer is $(1 - \lambda)c$. The value of $\lambda$ is exogenously given and dictates the degree of asymmetry in the roles of the active and passive agents.

The interpretation given to the network determines how benefits are obtained from the creation of links. Early studies on the internal evolution of networks focused on situations where the network simply describes the possibilities for transmission of valuable information from one individual to another. In these cases, the network evolves taking into account the incentives of individuals to form or sever links in order to obtain more information (e.g., Jackson and Wolinsky, 1996; Bala and Goyal, 2000). Later publications (e.g., Jackson and

Watts, 2002; Droste et al., 2000; Goyal and Vega-Redondo, 2004 and Bramoullé et al., 2004) have analyzed more elaborated frameworks where an agent plays a bilateral game with each of her "neighbors" (directly connected agents). Thus, apart from the decision over the links to form, an agent must decide the action taken in the accompanying game and rewards from different actions depend crucially on the actions chosen by neighbors. In the present paper, we have followed this last approach and studied the influence of the network structure on individual's behavior in the context of $2 \times 2$ anti-coordination games, i.e. games where a player's best response is to behave differently than the opponent.

The main contributions of this paper are two fold. On the one hand, it presents a non-cooperative model of network formation (i.e., standard Nash equilibrium concept applies) to describe link formation settings where generally both agents involved in the link bare part of its cost. This contrasts with previous papers in which the so called *pairwise stability* concept is used (see e.g., Jackson and Wolinsky, 1996; Jackson and Watts, 2002).[1] On the other hand, this paper analyzes network formation in the context of anti-coordination games meanwhile most related literature focuses on coordination games.[2]

The results of the paper can be summarized as follows. We provide a characterization of the Nash equilibria of the social game induced and show how this depends on the cost of the link $c$ and the cost share $\lambda$. We generally find that, as the cost increases the equilibrium networks become more sparse going from the complete to the empty network through several, more complex, intermediary network architectures. In addition, the cost has a profound impact on the proportion of players choosing the two actions in the anti-coordination game. When the cost is low, there is a unique proportion of players choosing each action which roughly corresponds with the proportion that would arise in the mixed strategy Nash equilibrium of the two person anti-coordination game. For higher values of the cost, we typically find a wider range of proportions sustained in equilibrium. We show how this range evolves as $c$ increases, stating the dependence on the value of $\lambda$. In addition, if the cost is considered as fixed and we vary the value of the cost share, we find that as the model becomes closer to a one-sided model (i.e., the agent proposing the link incurs in the whole cost of it) the range of proportions sustained in equilibrium increases. In other words, the higher the difference in the cost incurred by the active and passive agent in the link, the higher the multiplicity in the proportions of agents choosing each action in equilibrium. The intuition behind this result is the following. The higher the asymmetries in the roles of active and passive links, the more we can use the direction of links to sustain a variety of proportions as equilibrium

[1] Pairwise stability has some disadvantages in this context since agents cannot simultaneously change more than one component of their strategy. In particular, this rules out the possibility that an agent might decide to change her links precisely because she is also changing her action in the game.

[2] The sole exception is the work by Bramoulle et al. (2004) where anti-coordination games were first analyzed but with a one-sided model of network formation.

since an agent may be induced to choose an action that is relatively popular, because in equilibrium agents choosing the other action are actively forming all the links with her.[3] In fact, in one-sided mechanisms ($\lambda = 1$) this range is the highest possible whereas in two-sided mechanisms ($\lambda = \frac{1}{2}$) this range is the smallest. Specifically, in the latter case, there is a unique proportion sustained in equilibrium.

Among the equilibria that exist in our model, we pay special attention to those sharing a common feature. These are the *distribution insensitive* states. We say that a state is distribution insensitive if it is a Nash equilibrium for any possible distribution of active and passive "bidirectional" links, i.e. links that could be supported actively by either player involved in it. We show that, for all values of $c$ and $\lambda$, there exists a distribution insensitive state. Moreover, when the cost is not too high, distribution insensitive states represent a small subset of the whole set of Nash equilibria and thus it can be considered as a reasonable argument for equilibrium selection.

To conclude, we have addressed the issue of efficiency. The tension between efficiency and equilibrium originally highlighted by Jackson and Wolinsky (1996) is also present in this setting. Nevertheless, it is worth noting that, when considering symmetric anti-coordination games, we find that distribution insensitive states are the equilibria with the highest welfare.

The rest of the paper is organized as follows. The model is introduced in Section 2. The description of the Nash equilibria of the game as well as the distribution insensitive states are presented in Section 3. Section 4 deals with the comparison between efficiency and stability. Finally, Section 5 concludes. Some proofs have been relegated to the Appendix.

# 2 The model

Let $N = \{1, 2, \ldots, n\}$ be a set of players where $n \geq 2$. We are interested in modeling a situation where each of these players can choose the subset of other players with whom to interact via a fixed bilateral game. More precisely, the interaction between any two linked players is given by a $2 \times 2$ symmetric anti-coordination game with the common set of actions $A = \{\alpha, \beta\}$. For each pair of actions $a, a' \in A$, the payoff $\pi(a, a')$ earned by a player choosing $a$ when her partner plays $a'$ is given by the following table:

| 1 \ 2 | $\alpha$ | $\beta$ |
|---|---|---|
| $\alpha$ | $d$ | $e$ |
| $\beta$ | $f$ | $b$ |

[3] Notice that, in anti-coordination games as opposed to coordination games, agents prefer choosing the less popular action.

Table I: Payoff Table

This payoff table describes an anti-coordination game (i.e. an agent prefers to behave differently to her opponent) with two pure strategy equilibria, $(\alpha, \beta)$ and $(\beta, \alpha)$. In other words, we consider the following restrictions on the payoffs:

$$d < f \text{ and } b < e \tag{1}$$

We shall also assume that every player $i$ is obliged to choose the same action in the (generally) several bilateral games that she is engaged in. This assumption is natural in the present context; if players were allowed to choose a different action for every two-person game this would make the behavior of players in any particular game insensitive to the network structure.

Given an agent $i \in N$, she can make proposals to other agents in the population to form a link. Formally, let $g_i^p = (g_{i1}^p, g_{i2}^p, ..., g_{in}^p)$ be the set of proposals of agent $i$. We suppose that $g_{ij}^p \in \{0, 1\}$ and $g_{ij}^p = 1$ if $i$ has proposed to form a link with $j$ and $g_{ij}^p = 0$ otherwise. The profile $(g_1^p, g_2^p, ..., g_n^p)$ generates the *directed network of proposals* denoted by $g^p$ hereafter. The strategy space of player $i$ can be identified with $S_i = \mathcal{G}_i^p \times A$, where $\mathcal{G}_i^p$ is the set of her proposals and $A$ is the common action space of the underlying bilateral game.[4]

There exists a link between two agents in the population if at least one of them proposes it and the other one is willing to accept the offer. We refer to the proposer as the *active agent* and to the receiver of the proposal as the *passive agent*. Links are assumed costly; and specifically, the minimum cost required to form a link is $c > 0$. The active agent of the link incurs in a sunk cost of $\lambda c$ where $\lambda \in [1/2, 1]$, whereas the cost incurred by the passive agent in case of accepting this offer is $(1 - \lambda)c$. The value of $\lambda$ is exogenously given throughout the paper. Figure 1 provides a description of the link formation process. The acceptance of a link is not modeled explicitly as part of a second stage of the game. Instead, we incorporate in the model the assumption that, a passive agent will response optimally to the proposer's offer. Formally, consider agents $i, j \in N$ then, a link between them is formed if and only if one of the following conditions hold:

- Both agents are active, i.e. $\min\{g_{ij}^p, g_{ji}^p\} = 1$
- One of the agents (say $i$) is active and the other one gets a non-negative net payoff from the link. Formally,

$$g_{ij}^p = 1,\ g_{ji}^p = 0 \text{ and } \pi(a_j, a_i) - (1 - \lambda)c \geq 0$$

[4] In our formulation, players choose proposals and actions simultaneously. A first glance to the sequential counterparts of the model (actions then links or vice-versa) shows that the set of equilibrium outcomes would enlarge.

Insert Figure 1 about here

Given $(g^p, (a_i)_{i \in N})$ a network of proposals and specific profile of actions, we define the *network of directed links* (denoted by $g$) as the corresponding graph in which all proposals that were not accepted are deleted. Formally, $g = (g_1, ..., g_n)$, where $g_i = (g_{i1}, ..., g_{in})$ represents the set of links proposed by $i$ that actually formed. That is, $g_{ij} \in \{0, 1\}$ where $g_{ij} = 1$ if and only if agent $i$ has proposed the link with $j$ and either $j$ has also proposed the link with $i$ or $i$'s proposal is accepted by $j$.

For the sake of completeness, we denote by $\overline{g}$ the undirected graph resulting from $g$. Formally, $\overline{g} = (\overline{g}_1, ..., \overline{g}_n)$ where for each $i \in N$, $\overline{g_i} = (\overline{g}_{i1}, ..., \overline{g}_{in})$ represents the set of agents with whom $i$ plays the anti-coordination game. That is, $\overline{g}_{ij} \in \{0, 1\}$ where $\overline{g}_{ij} = \max\{g_{ij}, g_{ji}\}$.

In order to define the payoff function of the social game we need some additional notation. Let $N(i; g^p) = \{j \in N \text{ s.t. } g^p_{ij} = 1\}$ be the set of agents to whom $i$ has proposed a link and denote by $v(i; g^p)$ its cardinality. Similarly, let $N(i; g) = \{j \in N \text{ s.t. } g_{ij} = 1\}$ be the set of agents that accepted links proposed by $i$ and denote by $v(i; g)$ its cardinality. Finally, denote by $N(i; \bar{g}) = \{j \in N \text{ s.t. } \overline{g}_{ij} = 1\}$ to the set of agents with whom player $i$ plays the anti-coordination game, while $v(i; \bar{g})$ is the cardinality of this set. It is straightforward to see that the following inclusions hold:

$$N(i; g) \subseteq N(i; g^p)$$

and

$$N(i; g) \subseteq N(i; \bar{g})$$

Notice that, in general there is no inclusion between the sets $N(i; g^p)$ and $N(i; \bar{g})$.

In the setup being considered, the payoff of a player $i$ from playing some strategy $s_i = (g^p_i, a_i)$ when the strategies of other players are given by $s_{-i} = (s_1, s_2, ..s_{i-1}, s_{i+1} ..., s_n)$ can be written as follows:

$$\Pi_i(s_i, s_{-i}) = \sum_{j \in N(i; \bar{g})} \pi(a_i, a_j) - \nu(i; g^p) \cdot \lambda c - (\nu(i; \overline{g}) - \nu(i; g)) \cdot (1 - \lambda) c \qquad (2)$$

where $g$ and $\overline{g}$ are determined as a consequence of $s = (g^p, (a_i)_{i \in N})$.

Individual payoffs are aggregated across all the games played. Moreover, a player's cost is computed as the sum of the costs incurred from all the links she proposes plus the cost of those links she accepts. In our framework, the number of games an individual plays is endogenous, and we want to explicitly account for the influence of the size of the neighborhood. This motivates the aggregate formulation.

The above payoff expression allows us to particularize the standard notion of Nash equilibrium as follows. A strategy profile $s^* = (s_1^*, \ldots s_n^*)$ is said to be a *Nash equilibrium* for the game if, for all $i \in N$,

$$\Pi_i(s_i^*, s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*), \forall s_i \in S_i. \tag{3}$$

A Nash equilibrium is said to be strict if every player gets a strictly higher payoff with her current strategy than she would with any other strategy.

## 3 Analysis

In this section we analyze the set of strict Nash equilibria of the social game. We describe first the structure of Nash networks providing a complete characterization of the type of links that form. Second, we calculate the range of possible values for the number of agents playing each action ($\alpha$ or $\beta$) in equilibrium. To understand the main insights of the model, we focus on two different comparative statics exercises: (i) Consider $\lambda$ as a fixed parameter and vary $c$ (ii) Consider $c$ as a fixed parameter and vary $\lambda$.

Anti-coordination games have different possible payoffs configurations and we will see that they also lead to different types of Nash networks. By definition, we have $d < f$ and $e > b$. Without loss of generality, assume that

$$f \geq e$$

In other words, $\beta$-players (i.e., players who choose action $\beta$ in the anti-coordination game) earn a higher payoff than $\alpha$-players (i.e. players who choose action $\alpha$ in the anti-coordination game) in equilibrium. There are three possible payoffs ordering.

$$\begin{aligned} \text{Case 1} &: \quad b < e < d < f \\ \text{Case 2} &: \quad b \leq d < e \leq f \\ \text{Case 3} &: \quad d \leq b < e \leq f \end{aligned}$$

Each ordering corresponds to a different type of anti-coordination game. In Case 1, the payoff of coordinating on $\alpha$ is higher than the payoff of an $\alpha$-player in equilibrium. Therefore, Case 1 represents exploitation games akin to the Hawk-Dove game. In Cases 2 and 3, equilibrium payoffs are higher than any other payoffs. Cases 2 and 3 represent situations of pure complementary, in which both players earn higher payoffs at equilibrium than out of it. In Case 2 the payoff of coordinating on $\alpha$ is higher than the payoff of coordinating on $\beta$, while the situation is reversed in Case 3.

It is worth noting that Nash networks are *essential*. In other words, $g^p_{ij} = 1 \Rightarrow g^p_{ji} = 0$ in equilibrium.[5] On the other hand, their structure depends on how $c$ and $\lambda$ compare with the parameters of the game. For example, when $\lambda c > b$ (i.e. the cost of proposing a link is higher than the payoff obtained when both agents play $\beta$), $\beta$-players do not have an incentive to form links with other $\beta$-players. Therefore, in equilibrium there is no link among $\beta$-players. Instead, when $\lambda c < b$, $\beta$-players are willing to propose links with any other agent playing $\beta$. In addition, passive $\beta$-players are also willing to accept these offers since $\lambda \in [\frac{1}{2}, 1]$ and thus $(1-\lambda)c < b$. Thus, in equilibrium, all $\beta$-players are directly linked with all other $\beta$-players and the network of links among them is essential and complete. The argument is similar for any other type of link. For example, if $\lambda c > f$, there is no link proposed from $\beta$-players to $\alpha$-players. If the contrary holds, i.e. $\lambda c < f$, all $\beta$-players would want to propose links with all $\alpha$-players. These links would form, however, only if the $\alpha$-players are willing to accept these offers, i.e. $(1-\lambda)c < e$.

The following shorthand notation will allow us to refer to all the possible types of Nash networks. This is a qualitative representation of the network where we simply specify the type of links that are profitable, i.e. that will form in equilibrium (if an equilibrium actually exists). Here, "to be linked to" is taken to mean that the links go in only one direction, whereas "to be linked with" signifies that the links may go in either direction - only in one of them of course, since equilibrium networks involve no redundant links. This type of links will be referred as *bidirectional links* since the two agents involved can afford the cost of proposing it. A formal definition, however, will be presented later in the paper.

- $\beta \; \emptyset \; \alpha$ : the *empty network*.
- $\beta \rightarrow \alpha$ : all $\beta$-players are linked to all $\alpha$-players, but no $\alpha$-player is linked to a $\beta$-player.
- $\beta \rightleftharpoons \alpha$ : all $\beta$-players are linked with all $\alpha$-players.
- $\beta \rightarrow \vec{\alpha}$ : all $\beta$-players are linked to all $\alpha$-players, and all $\alpha$-players are linked with all $\alpha$-players.
- $\beta \rightleftharpoons \vec{\alpha}$ : all $\alpha$-players are linked with all $\alpha$-players and with all $\beta$-players.
- $\vec{\beta} \rightleftharpoons \alpha$ : all $\beta$-players are linked with all $\beta$-players and with all $\alpha$-players.
- $\vec{\alpha}$ : all $\alpha$-players are linked with all $\alpha$-players.
- $\vec{\beta} \rightleftharpoons \vec{\alpha}$ : the *complete network*.

[5] The sole exception occurs if $\lambda = \frac{1}{2}$ where agents feel indifferent between being the active or passive agent in the link.

The graphs $\beta \rightarrow \alpha$ and $\beta \rightleftharpoons \alpha$ are referred as *bipartite networks* because only links across groups (i.e., between $\alpha$-players and $\beta$-players) are formed, while $\beta \rightarrow \vec{\alpha}$, $\beta \rightleftharpoons \vec{\alpha}$ and $\vec{\beta} \rightleftharpoons \alpha$ are referred as *semi-bipartite networks* since in addition to links across groups, links between agents choosing one particular action also exist.

## 3.1 Varying the overall cost of the link

As a first approach, we consider the cost share as fixed and analyze the results when the cost of links varies. Using the above notation, the following result describes how the parameters of the model determine the type of Nash network.

**Proposition 1** *If there exists a strict Nash equilibrium, its network structure exhibits the following pattern of link formation:*

| *Exploitation games* | | | |
|---|---|---|---|
| *Case 1.1* | | *Case 1.2* | |
| $0 < c < \frac{b}{\lambda}$ | $\vec{\beta} \rightleftharpoons \vec{\alpha}$ | $0 < c < \frac{b}{\lambda}$ | $\vec{\beta} \rightleftharpoons \vec{\alpha}$ |
| $\frac{b}{\lambda} < c < \frac{e}{\lambda}$ | $\beta \rightleftharpoons \vec{\alpha}$ | $\frac{b}{\lambda} < c < \frac{e}{\lambda}$ | $\beta \rightleftharpoons \vec{\alpha}$ |
| $\frac{e}{\lambda} < c < \frac{e}{1-\lambda}$ | $\beta \rightarrow \vec{\alpha}$ | $\frac{e}{\lambda} < c < \frac{d}{\lambda}$ | $\beta \rightarrow \vec{\alpha}$ |
| $\frac{e}{1-\lambda} < c < \frac{d}{\lambda}$ | $\vec{\alpha}$ | $\frac{d}{\lambda} < c < \min\{\frac{e}{1-\lambda}, \frac{f}{\lambda}\}$ | $\beta \rightarrow \alpha$ |
| $\frac{d}{\lambda} < c$ | $\beta \ \emptyset \ \alpha$ | $\min\{\frac{e}{1-\lambda}, \frac{f}{\lambda}\} < c$ | $\beta \ \emptyset \ \alpha$ |

| *Complementarity games* | | | |
|---|---|---|---|
| *Case 2* | | *Case 3* | |
| $0 < c < \frac{b}{\lambda}$ | $\vec{\beta} \rightleftharpoons \vec{\alpha}$ | $0 < c < \frac{d}{\lambda}$ | $\vec{\beta} \rightleftharpoons \vec{\alpha}$ |
| $\frac{b}{\lambda} < c < \frac{d}{\lambda}$ | $\beta \rightleftharpoons \vec{\alpha}$ | $\frac{d}{\lambda} < c < \frac{b}{\lambda}$ | $\vec{\beta} \rightleftharpoons \alpha$ |
| $\frac{d}{\lambda} < c < \frac{e}{\lambda}$ | $\beta \rightleftharpoons \alpha$ | $\frac{b}{\lambda} < c < \frac{e}{\lambda}$ | $\beta \rightleftharpoons \alpha$ |
| $\frac{e}{\lambda} < c < \min\{\frac{e}{1-\lambda}, \frac{f}{\lambda}\}$ | $\beta \rightarrow \alpha$ | $\frac{e}{\lambda} < c < \min\{\frac{e}{1-\lambda}, \frac{f}{\lambda}\}$ | $\beta \rightarrow \alpha$ |
| $\min\{\frac{e}{1-\lambda}, \frac{f}{\lambda}\} < c$ | $\beta \ \emptyset \ \alpha$ | $\min\{\frac{e}{1-\lambda}, \frac{f}{\lambda}\} < c$ | $\beta \ \emptyset \ \alpha$ |

Insert Figure 2 about here

The proof is straightforward and thus omitted. Several interesting points follow from the above result. First, it shows that (except for very low costs), the nature of links is quite complicated and the link proposal, and hence the network architecture, depends very much on the game that is being played.

There are two types of exploitation games. The first type (Case 1.1) holds when $\lambda < \frac{d}{d+e}$ and is characterized by the fact that, for a certain range of the cost (specifically, $\frac{1}{1-\lambda}e < c < \frac{1}{\lambda}d$) the only "profitable" links are those between two $\alpha$-players. The second type (Case 1.2) holds when $\lambda > \frac{d}{d+e}$ and it departs from the other type by offering a bipartite graph as a Nash architecture (for the range of costs $\frac{d}{\lambda} < c < \min\{\frac{e}{1-\lambda}, \frac{f}{\lambda}\}$).

If the game is one of strict complementarity (as in Cases 2 and 3), for certain values of the cost, it supports bipartite graphs $\beta \rightleftharpoons \alpha$ as Nash networks. That is, both $\alpha$-players and $\beta$-players have an interest to be linked to players choosing the other action, while they do not wish to be linked with players choosing the same action.

A second point worth noting concerns the effect of increasing the linking costs. In each of the three types of anti-coordination games, the effect of higher costs is broadly similar. The payoffs of the anti-coordination game as well as $\lambda$ define cut-off values such that, as the costs of link proposal surpasses them, an economic opportunity disappears along with its corresponding type of link. The lengths of these cost ranges depend crucially on the value of $\lambda$. For example, in Case 2 the values of the cost for which we obtain a complete network are $c < b$ if $\lambda = 1$ whereas it spans to $c < 2b$ if $\lambda = \frac{1}{2}$. This is because in the former case, the cost of the link is incurred only by the active agent whereas in the later case it is divided equally between both agents involved in the link. Thus, higher values of the cost make the link still profitable. The situation is similar for any other type of network. For example, the values of the cost for which we obtain a semi-bipartite network of the type $\beta \rightleftharpoons \vec{\alpha}$ are $b < c < d$ if $\lambda = 1$ whereas they are $2b < c < 2d$ if $\lambda = \frac{1}{2}$. Notice that, if $d < 2b$ these two ranges for the cost are disjoint. In general, we find that, as the cost of link formation rises, the possible types of Nash networks become more sparse, going from the complete network to the empty network through three intermediary cases.[6]

We now analyze for every given value of $\lambda$, how the number of players choosing each action in equilibrium depends on $c$. In order to do this we restrict our attention to a particular class of anti-coordination games, those that satisfy the following condition:

$$2d < f + e \tag{4}$$

This always holds for complementarity games, i.e. Cases 2 and 3, but it imposes an additional restriction for exploitation games, i.e. Case 1. Notice that, if condition (4) does not hold this represents an extreme case of exploitation game where the efficiency of links between $\alpha$ and $\beta$-players is lower than the efficiency of links between $\alpha$-players.

Let $s$ be any given strategy profile, and denote by $n_k^s$ to the number of $k$-players in it, where $k = \alpha, \beta$. Our next result derives the lower and upper bounds for $n_\alpha^s$ and $n_\beta^s$ in equilibrium. We derive this result by examining the best-responses for every possible case. To do so, we need a piece of notation. Denote $p_\beta = \frac{f-d}{f-d+e-b}$. Notice that $p_\beta$ is the probability of playing

[6] An exception occurs in Case 1.1 where, for a cost sufficiently high, there is an abrupt transition from a complete network with all agents choosing $\alpha$ ( i.e. $\vec{\alpha}$) to the empty network.

$\beta$ in the mixed strategy equilibrium of the anti-coordination game. Fix $\lambda \in [\frac{1}{2}, 1]$ and define the two following auxiliary functions:

$$\psi_\lambda(c) = \begin{cases} p_\beta \text{ if } c \leq \min\{\frac{1}{\lambda}d, \frac{1}{1-\lambda}b\} \\ \frac{f-\lambda c}{f+e-b-\lambda c} \text{ if } \frac{1}{\lambda}d < c \leq \min\{\frac{1}{1-\lambda}b, \frac{1}{\lambda}f\} \\ \frac{f-d}{f+e-d-(1-\lambda)c} \text{ if } \frac{1}{1-\lambda}b < c \leq \min\{\frac{1}{\lambda}d, \frac{1}{1-\lambda}e\} \\ \frac{f-\lambda c}{f+e-c} \text{ if } \max\{\frac{1}{1-\lambda}b, \frac{1}{\lambda}d\} < c \leq \min\{\frac{1}{\lambda}f, \frac{1}{1-\lambda}e\} \end{cases}$$

and

$$\varphi_\lambda(c) = \begin{cases} p_\beta \text{ if } c \leq \min\{\frac{1}{\lambda}b, \frac{1}{1-\lambda}d\} \\ \frac{f-d}{f+e-d-\lambda c} \text{ if } \frac{1}{\lambda}b < c \leq \min\{\frac{1}{1-\lambda}d, \frac{1}{\lambda}e\} \\ \frac{f-(1-\lambda)c}{f+e-b-(1-\lambda)c} \text{ if } \frac{1}{1-\lambda}d < c \leq \frac{1}{\lambda}b \\ \frac{f-(1-\lambda)c}{f+e-c} \text{ if } \max\{\frac{1}{\lambda}b, \frac{1}{1-\lambda}d\} < c \leq \frac{1}{\lambda}e \\ 1 \text{ if } \frac{1}{\lambda}e < c \leq \min\{\frac{1}{\lambda}f, \frac{1}{1-\lambda}e\} \end{cases}$$

Note that $\varphi_\lambda(c)$ and $\psi_\lambda(c)$ are continuous. In the Appendix we show that $\psi_\lambda(c) \leq \varphi_\lambda(c)$ for all values of $c < \min\{\frac{1}{\lambda}f, \frac{1}{1-\lambda}e\}$ and $\lambda \in [\frac{1}{2}, 1]$. These functions bound the relative sizes of the different $\alpha$- and $\beta$-parts of the network, as established by the following result.

**Proposition 2** *Assume $2d < f + e$. The following statements hold:*

*If $c \leq \min\{\frac{1}{1-\lambda}e, \frac{1}{\lambda}f\}$ there exists a strict Nash equilibrium with $n_\beta$ individuals doing $\beta$ if and only if*

$$(n-1)\psi_\lambda(c) < n_\beta < (n-1)\varphi_\lambda(c) + 1$$

*If $c > \min\{\frac{1}{1-\lambda}e, \frac{1}{\lambda}f\}$ there is no strict Nash equilibrium*

The proof can be found in the Appendix.

Several interesting points follow from this result. It provides the precise relationship between $c$ and the range of proportions $\frac{n_\beta}{n_\alpha}$ sustained in equilibrium in the respective games. In particular, it states that for a low cost of forming links, the proportion of players choosing actions $\alpha$ and $\beta$ corresponds (roughly) to the mixed-strategy Nash equilibrium of the two-person anti-coordination game. This simply follows from the fact that, for low linking costs, players have incentives to form the complete network and hence the link formation mechanism has no particular influence on individual behavior. However, beyond this low range, $c$ has a profound impact on individual choice of actions which depends also on the value of $\lambda$.

If $\lambda$ is sufficiently high the results resemble those obtained by Bramoullé et all (2004) for $\lambda = 1$. The upper bound $\varphi_\lambda(c)$ (weakly) increases with respect to $c$ whereas the lower bound $\psi_\lambda(c)$ (weakly) decreases. This implies not only that the set of proportions sustained in equilibrium increases in parallel with $c$ but also that these sets are contained one in another. To illustrate this, focus on Case 2 of the anti-coordination game depicted in Figure 3. For the sake of concreteness consider the cost range $\frac{1}{\lambda}b < c < \frac{1}{\lambda}d$ where the upper bound $\varphi_\lambda(c)$ is increasing whereas the lower bound $\psi_\lambda(c)$ is constant. An intuition of why this is so is the following: in this range, Nash equilibria are semi-bipartite networks, i.e. $\beta \rightleftharpoons \vec{\alpha}$. As a general statement -applicable to all other cases- we know that, the lower bound for $n_\beta$ is obtained imposing that all links between $\alpha$ and $\beta$-players are proposed by the $\beta$-players (i.e. $\beta \rightharpoonup \vec{\alpha}$). The reason being that this distribution of links maximizes the incentive of an $\alpha$-player to maintain her action and thus sustains lower values for $n_\beta$ in equilibrium. What happens with the incentives of remaining as an $\alpha$-player as we increase $c$? Notice that if an $\alpha$-player switches to $\beta$ the structure of the network will remain intact since she will maintain all her "old" passive links with the rest of the $\beta$-players given that they are very cheap. Consequently, an increase in the cost of links will not affect her incentives to switch actions and thus the lower bound for $n_\beta$ in this range remains constant. For analogous reasons, the upper bound for $n_\beta$ is obtained imposing that all links between $\alpha$ and $\beta$-players are proposed by the $\alpha$-players (i.e. $\beta \leftharpoonup \vec{\alpha}$). As before, the reason for this is that this distribution of links maximizes the incentive of an $\beta$-player to maintain her action. Again, we can formulate the following question: What happens with the incentives of remaining as a $\beta$-player as we increase $c$? Notice that, if a $\beta$-player switches to $\alpha$ the network will definitely change since this player would have to form actively links with all the $\beta$-players. Therefore, an increase in the cost of the link would decrease her incentives to switch which implies that higher values for $n_\beta$ are sustained in equilibrium. Consequently the upper bound for $n_\beta$ increases.

For intermediate values of $\lambda$, the bounds for $n_\beta$ in equilibrium exhibit a different behavior. In contrast with the previous case, there are some ranges of the cost where the upper bound $\varphi_\lambda(c)$ decreases with respect to $c$ and others where the lower bound $\psi_\lambda(c)$ increases. Hence, as $c$ increases the set of values for $n_\beta$ in equilibrium does not have to include the set obtained for smaller values of $c$. To illustrate this, focus on Case 2 of the anti-coordination game depicted in Figure 4. Notice that, when the cost is in the range $\frac{1}{1-\lambda}b < c < \frac{1}{\lambda}d$, the lower bound $\psi_\lambda(c)$ increases with respect to $c$. The intuition for this is as follows: in this range, Nash equilibria are semi-bipartite networks, i.e. $\beta \rightleftharpoons \vec{\alpha}$. Thus, the lower bound for $n_\beta$ is obtained imposing that all links between $\alpha$ and $\beta$-players are proposed by the $\beta$-players. Nevertheless, passive links are also costly and therefore an $\alpha$-player has to incur in a cost of $(1-\lambda)c$ for each passive link. Since $b < (1-\lambda)c$, if an $\alpha$-player considers the possibility of switching to $\beta$, she will not accept to interact with any of the other $\beta$-players and therefore would refuse to sustain any passive link with a $\beta$-player. The rest of the network however, would remain the same. To sum up, when an $\alpha$-player switches to action $\beta$ she is saving $n_\beta(1-\lambda)c$. Hence, if the value of $c$ increases, the savings in the case of switching to $\beta$

also increases. Therefore, in contrast with the previous case, as $c$ increases, an $\alpha$ player has higher incentives to switch to action $\beta$ which implies that the lower bound for $n_\beta$ increases.

Finally, let us assume that $\lambda$ is low. For the sake of concreteness, assume $\lambda = \frac{1}{2}$. In this case, the linking cost is divided equally between the active and passive agent and thus, there is no advantage from being the passive agent in the link. This generally implies that the distribution of links has no influence on the incentives to switch actions. Therefore, in this setting, when a player chooses her best response she only takes into consideration the relative sizes of the groups of agents choosing each action. Consequently, there exists a unique proportion $(\frac{n_\beta}{n_\alpha})$ sustained in equilibrium (for an illustrative example see Figure 5).

Insert Figures 3, 4 and 5 about here

## 3.2 Varying the cost share of the link

In the previous section, we have analyzed the model considering the cost share $\lambda$ as a fixed parameter and studying how the results change when varying the value of $c$. To explicitly account for the influence of the cost share $\lambda$ in the equilibrium outcomes, we assume $c$ is fixed, and analyze how the results change as we vary $\lambda$.

We now analyze for every given value of $c$, how the number of players choosing each action in equilibrium depends on $\lambda$. To this effect, it is useful to introduce two auxiliary functions $\varphi_c(\lambda)$ and $\psi_c(\lambda)$ as follows:

$$\psi_c(\lambda) = \begin{cases} p_\beta \text{ if } max\{\frac{1}{2}, 1-\frac{1}{c}b\} < \lambda \leq \min\{\frac{1}{c}d, 1\} \\ \frac{f-\lambda c}{f+e-b-\lambda c} \text{ if } max\{\frac{1}{2}, \frac{1}{c}d, 1-\frac{1}{c}b\} < \lambda \leq \min\{\frac{1}{c}f, 1\} \\ \frac{f-d}{f+e-d-(1-\lambda)c} \text{ if } max\{\frac{1}{2}, 1-\frac{1}{c}e\} < \lambda \leq \min\{\frac{1}{c}d, 1-\frac{1}{c}b\} \\ \frac{f-\lambda c}{f+e-c} \text{ if } \max\{\frac{1}{2}, 1-\frac{1}{c}e, \frac{1}{c}d\} < \lambda \leq \min\{\frac{1}{c}f, 1-\frac{1}{c}b\} \end{cases}$$

and the function,

$$\varphi_c(\lambda) = \begin{cases} p_\beta \text{ if } max\{\frac{1}{2}, 1-\frac{1}{c}d\} < \lambda \leq \min\{\frac{1}{c}b, 1\} \\ \frac{f-d}{f+e-d-\lambda c} \text{ if } max\{\frac{1}{2}, \frac{1}{c}b, 1-\frac{1}{c}d\} < \lambda \leq \min\{\frac{1}{c}e, 1\} \\ \frac{f-(1-\lambda)c}{f+e-b-(1-\lambda)c} \text{ if } \frac{1}{2} < \lambda \leq \min\{\frac{1}{c}b, 1-\frac{1}{c}d\} \\ \frac{f-(1-\lambda)c}{f+e-c} \text{ if } max\{\frac{1}{2}, \frac{1}{c}b\} < \lambda \leq \min\{1-\frac{1}{c}d, \frac{1}{c}e\} \\ 1 \text{ if } max\{\frac{1}{c}e, 1-\frac{1}{c}e\} < \lambda \leq \min\{\frac{1}{c}f, 1\} \end{cases}$$

Notice that $\varphi_c(\lambda)$ and $\psi_c(\lambda)$ are the same functions than $\varphi_\lambda(c)$ and $\psi_\lambda(c)$ but the former ones are stated in terms of $\lambda$ whereas the later ones are stated in terms of $c$. Note that, for all $c \geq 0$, $\psi_c(\lambda)$ is decreasing whereas $\varphi_c(\lambda)$ is increasing. These functions bound the relative sizes of the different $\alpha$- and $\beta$-parts of the network, as established by the following result.

**Proposition 3** *Assume $2d < f + e$. The following statements hold:*

*(i) If $max\{\frac{1}{2}, 1 - \frac{1}{c}e\} \leq \lambda \leq \min\{\frac{1}{c}f, 1\}$ there exists a strict Nash equilibrium with $n_\beta$ individuals doing $\beta$ if and only if*

$$(n-1)\psi_c(\lambda) < n_\beta < (n-1)\varphi_c(\lambda) + 1$$

*(ii) If $\frac{1}{2} \leq \lambda < \max\{\frac{1}{2}, 1 - \frac{1}{c}e\}$ or $\frac{1}{c}f < \lambda \leq 1$ there is no strict Nash equilibrium.*

The proof is analogous to that of Proposition 2 and thus will be presented in the Appendix as well. We observe that, the higher the value of $\lambda$ the larger the set of proportions $\frac{n_\beta}{n_\alpha}$ sustained in equilibrium. The intuition of this result is as follows: The higher $\lambda$, the higher the difference in the cost incurred by the active and passive agent from the link and thus the direction of the links influences more the incentives of agents. As a consequence, some proportions are sustained in equilibrium only because of a particular distribution of active and passive links. In general, the lower the size of a group of agents choosing a particular action, the higher the number of links proposed by them to agents from the other group.

For low values of $\lambda$, we can no longer count on these arguments in order to sustain a wide variety of proportion in equilibrium and therefore the set of Nash equilibria shrinks. Indeed, as aforementioned, for $\lambda = \frac{1}{2}$ we have a unique equilibrium value for $n_\beta$ (an example is depicted in Figure 6).

Insert Figure 6 about here

## 3.3 Distribution insensitive states

The above results provide the qualitative features of Nash networks as well as the proportions of players choosing each action sustained in equilibrium. However, they do not typically give information of either the distribution of active and passive links or the payoff distribution among the agents at equilibrium.

Among the typically multiple equilibria that exist in our model, we focus on those sharing a common feature. These are the *distribution insensitive* states. In order to define this concept formally we need to specify first the meaning of *bidirectional links.*

**Definition 4** *Given a state $(g^p, (a_i)_{i\in N})$, a link between two agents (say $i$ and $j$) is bidirectional if and only if the following conditions hold:*

$$\pi(a_i, a_j) - \lambda c \geq 0 \text{ and } \pi(a_j, a_i) - \lambda c \geq 0$$

In other words, both agents involved in the link should be willing to propose it. This leads us to the concept of distribution insensitive states.

**Definition 5** *A state $(g^p, (a_i)_{i\in N})$ is distribution insensitive if any state resulting from a redistribution of active and passive bidirectional links is a strict Nash equilibrium.*

Typically, the higher (lower) the number of $\beta$-players in equilibrium, the higher (lower) the number of $\alpha$-players that support actively their links. Nevertheless, if a state is distribution insensitive these considerations are not relevant since the allocation of costs of the links (i.e. distribution of active and passive links) does not affect equilibria. In other words, distribution insensitive states are Nash equilibria robust to changes in the direction of links.

Notice that, if a state $s$ is distribution insensitive then, any other state $s'$ differing from $s$ only in the distribution of active and passive bidirectional links will also be distribution insensitive.

Thus, a set of distribution insensitive states can be characterized by the corresponding proportion of agents choosing each action. In particular, we say that a specific number of agents $n_\beta^*$ choosing $\beta$ is *distribution insensitive* if there exists a certain state with this number of $\beta$-players which is distribution insensitive. The following result shows that for every $c \geq 0$ and any given value of $\lambda \in [1/2, 1]$, there exists a distribution insensitive $n_\beta^*$ .

**Proposition 6** *Let $\lambda \in [1/2, 1]$ and $c \geq 0$. The following statements hold:*

*(i)If the Nash network is of type $\vec{\beta} \rightleftharpoons \vec{\alpha}$ then $n_\beta^*$ is distribution insensitive iff*

$$(n-1)p_\beta \leq n_\beta^* \leq (n-1)p_\beta + 1$$

*(ii)If the Nash network is of type $\beta \rightleftharpoons \vec{\alpha}$ then $n_\beta^*$ is distribution insensitive iff*

$$(n-1)\frac{f-d}{f-d+e-\lambda c} \leq n_\beta^* \leq (n-1)\frac{f-d}{f-d+e-\lambda c} + 1$$

*(iii)If the Nash network is of type $\vec{\beta} \rightleftharpoons \alpha$, then $n_\beta^*$ is distribution insensitive iff*

$$(n-1)\frac{f-\lambda c}{f-b+e-\lambda c} \leq n_\beta^* \leq (n-1)\frac{f-\lambda c}{f-b+e-\lambda c} + 1$$

*(iv)If the Nash network is of type $\beta \rightleftharpoons \alpha$, then $n_\beta^*$ is distribution insensitive iff*

$$(n-1)\frac{f-\lambda c}{f+e-2\lambda c} \leq n_\beta^* \leq (n-1)\frac{f-\lambda c}{f+e-2\lambda c} + 1$$

*(v)For any other type of network, all Nash equilibria are distribution insensitive.*

This result implies that, there exists a *unique* distribution insensitive proportion in the cases $\vec{\beta} \rightleftharpoons \vec{\alpha}$, $\beta \rightleftharpoons \vec{\alpha}$, $\vec{\beta} \rightleftharpoons \alpha$ and $\beta \rightleftharpoons \alpha$ whereas in the cases $\beta \rightarrow \alpha$, $\beta \rightarrow \overrightarrow{\alpha}$ and $\overrightarrow{\alpha}$ all Nash equilibria are distribution insensitive. We find that, the relative density of agents choosing each action in the game generally depends on $c$ and $\lambda$. In particular, for $\beta \rightleftharpoons \vec{\alpha}$ and $\beta \rightleftharpoons \alpha$ ($\vec{\beta} \rightleftharpoons \alpha$) $n_\beta^*$ increases (decreases) as $c$ or $\lambda$ increase whereas it is constant for $\vec{\beta} \rightleftharpoons \vec{\alpha}$. To illustrate the selective power of this concept see the example depicted in Figure 7. Note that, in the example, if $b < c < e$ there exists a unique $n_\beta^*$ distribution insensitive among the multiple values of $n_\beta$ sustained in equilibrium.

Insert Figure 7 about here

We now present the essential argument for this result, focusing for concreteness on the range $(1/\lambda)\max\{b,d\} < c < (1/\lambda)e$, where equilibrium networks are bipartite. Let $s$ be any strategy profile. Moreover, let $q_i^{s,k}$ be the number of active links of player $i$ with players choosing action $k$, where $k \in \{\alpha, \beta\}$. We will avoid superscript $s$ if there is no possible confusion. Consider any distribution insensitive state with $n_\beta$ players choosing $\beta$. Let $i \in N$ be an agent who chooses $\alpha$ in the underlying state and supports $q_i^\beta$ links to $\beta$-players. Then, in order for this player to be choosing a best response, a necessary and sufficient condition is that

$$n_\beta e - \lambda c q_i^\beta - (1-\lambda)c(n_\beta - q_i^\beta) > (n - n_\beta - 1)(f - \lambda c) + R(c) \tag{5}$$

where $R(c) = (n_\beta - q_i^\beta)(b - (1-\lambda)c)$ if $c < \frac{1}{1-\lambda}b$ (i.e. passive links between two $\beta$-players are profitable) and $R(c) = 0$ otherwise. Notice that, in the former case a necessary and sufficient condition for player $i$ to be doing a best response is,

$$n_\beta > (n-1)\frac{f - \lambda c}{f - \lambda c + e - (1-\lambda)c} + q_i^\beta \frac{\lambda c - (1-\lambda)c}{f - \lambda c + e - (1-\lambda)c} \tag{6}$$

whereas in the latter case the condition is,

$$n_\beta > (n-1)\frac{f - \lambda c}{f - \lambda c + e - b} + q_i^\beta \frac{\lambda c - b}{f - \lambda c + e - b} \tag{7}$$

The right hand sides of expressions (6) and (7) are both increasing in $q_i^\beta$ and therefore they reach a maximum at $q_i^\beta = n_\beta$. Moreover, observe that, substituting $n_\beta$ for $q_i^\beta$ in both equations, the same condition is obtained given by:

$$n_\beta > (n-1)\frac{f - \lambda c}{f + e - 2\lambda c} \tag{8}$$

which is necessary and sufficient for distribution insensitivity to apply to the agent considered. Turning now the attention to the counterpart condition, for any agent $j$ choosing $\beta$, note that, we can argue by symmetry with the previous case and find that $j$ is choosing a best response if and only if,

$$n_\alpha > (n-1)\frac{e - \lambda c}{f + e - 2\lambda c}$$

thus,

$$n_\beta < (n-1)\frac{f - \lambda c}{f + e - 2\lambda c} + 1 \tag{9}$$

which is again a necessary and sufficient condition for distribution insensitivity concerning any player choosing $\beta$. Combining (8) and (9), the desired conclusion follows. The detailed proof of the remaining cases is relegated to the Appendix.

# 4 Equilibrium vs. efficiency

To conclude, we study the welfare properties of the various states and compare the concept of efficiency and stability (i.e., Nash equilibrium). There are many ways to define welfare. Here we identify welfare with the sum of individuals' payoffs. Specifically, the welfare of a strategy profile $s = (s_1, ..., s_n)$, denoted as $W(s)$ is set equal to the sum of the individuals' payoffs,

$$W(s) = \sum_{i=1}^{n} \Pi_i(s).$$

We say that a state $s$ is *efficient* if and only if $W(s) \geq W(s')$, for all $s' \in S$.

Given an efficient state, links cannot be redundant. In other words, only one of the two agents involved proposes.[7] Moreover, only those links satisfying that the aggregate benefit (i.e. $e+f$, $2d$ or $2b$) is higher than the overall cost of the link (i.e. $c$) are formed. Since this is independent of $\lambda$, the efficiency results obtained in the previous work by Bramoullé et al.(2004) for the one-sided model $\lambda = 1$ can be directly generalized to the case $\lambda \in [1/2, 1]$. We thus remit to this paper to find a formal description of the set of efficient states. The main results obtained in this respect are as follows. As the linking cost increases, efficient networks become less connected going from the complete network to the empty network through two intermediary cases. Moreover, efficiency generally selects a unique relative size of the two parts. The reason for this is precisely that in the welfare analysis active and passive links have *no roles* and thus, the distribution of active and passive links does not enhance multiplicity. In particular, when the efficient network is bipartite, the efficient profile is perfectly balanced for all values of the parameters (i.e., $n_\beta = n_\alpha = \frac{n}{2}$). The reason is that, when the efficient network is bipartite, each link provides the same welfare contribution $e+f-c$. Therefore, in order to maximize welfare the number of links must be maximized, which is obtained when the two groups of players have the same size.

We conclude that, in general, Nash profiles are not efficient and vice versa. The reason for this is two-fold:

First, the type of links formed in an equilibrium and efficient profile do not generally coincide. To illustrate this consider the case $\lambda = 1$ and assume $b < c < 2b$. Links between $\beta$-players are efficient since they increase welfare but they are not sustained in equilibrium given that the cost of proposing is higher than the corresponding benefit. However, this tension is present for any other value of $\lambda \in [\frac{1}{2}, 1]$ as well. For instance, if $\lambda = \frac{1}{2}$ and $\frac{c}{2} < e < c < f$ links between $\alpha$ and $\beta$-players are efficient and nevertheless they are not sustained in equilibrium since $\alpha$-players have no incentive to form links with $\beta$-players.

[7] Notice that, this is true with the sole exception of $\lambda = \frac{1}{2}$ in which case an efficient state can contain redundant links.

A second source of discrepancy arises from the difference in the proportion of players choosing each action in efficient and equilibrium profiles. This tension appears when the two actions in the anti-coordination game are asymmetric (e.g. $f > e$). To illustrate consider $\lambda = \frac{1}{2}$ and $2b, 2d < c < 2e, 2f$. Note that, in this case, both efficient and equilibrium networks are bipartite graphs. Nevertheless, $\frac{n_\beta}{n_\alpha} = \frac{1}{2}$ in the efficient states whereas $\frac{n_\beta}{n_\alpha} > \frac{1}{2}$ in equilibrium. In other words, the number of players choosing $\beta$ in equilibrium is higher than what would be collectively optimal. The intuition behind this result is straightforward. Since action $\beta$ in equilibrium provides higher benefits than $\alpha$, to compensate for this and make action $\alpha$ incentive compatible, the number of $\beta$-players must be higher.

We thus ask the following natural questions. Among the Nash equilibria, which profiles yield highest welfare? When does a Nash profile yield higher welfare than another one? In order to give an answer to these question, we must find the efficient values for $n^*_\beta$ with the restriction that only those links that are present in the Nash equilibria profile will be formed. Once obtaining the value $n^*_\beta$ we can argue that, the Nash profile with highest welfare is the profile whose $n_\beta$ is closest to $n^*_\beta$ . More generally, the closer $n_\beta$ is to $n^*_\beta$ the higher the welfare of the Nash profile. For instance, when $2b, 2d < c < \frac{1}{\lambda}e, \frac{1}{\lambda}f$, Nash and efficient networks are bipartite, and the Nash profile with highest welfare is the one closest to the efficient state, i.e. where the share of the population playing $\beta$ is closest to $1/2$. However, suppose that $\frac{1}{\lambda}b, \frac{1}{\lambda}d < c < \frac{1}{\lambda}e, \frac{1}{\lambda}f, 2b, 2d$. This implies that Nash networks are bipartite and efficient networks are complete. Since Nash networks are bipartite, the welfare of a Nash profile $s$ is simply $W(s) = (e + f)n_{\alpha\beta}$. Therefore, as well as before, the Nash profile that yield highest welfare is the one where the share of the population playing $\beta$ is closest to $1/2$. Note that in this example, the efficient share applying to Nash profiles is different from the efficient share applied to all states. This arises from the fact that, in this case, the links formed in the efficient and equilibrium networks differ.

Finally, if we consider the case of anti-coordination games where actions are symmetric, we find that distribution insensitiveness plays an important role in determining the most efficient states among the Nash equilibria as described in the following result.

**Proposition 7** *Suppose $f = e$ and $b = d$. The distribution insensitive states are the most efficient states among the Nash equilibria. Moreover, if $\lambda = 1/2$ the equilibrium and efficient states coincide.*

**Proof**: Using the general proof presented by Bramoullé et al. (2004) it is straightforward to show that the efficient states are complete networks with $n_\beta = n/2$ if $c < 2b$, bipartite networks with $n_\beta = n/2$ if $2b < c < 2f$ and empty networks if $2f < c$. On the other hand, the distribution insensitive states are complete networks with $n_\beta = n/2$ if $c < \frac{1}{\lambda}b$, bipartite networks with $n_\beta = n/2$ if $\frac{1}{\lambda}b < c < \frac{1}{\lambda}f$ and empty networks if $\frac{1}{\lambda}f < c$. A comparison between the efficient and distribution insensitive states shows that the distribution insensitive states either coincide with the efficient states (for example when $c < \frac{1}{\lambda}b$) or they are the most efficient states among the Nash equilibria (for example when $\frac{1}{\lambda}b < c < \frac{1}{\lambda}b$). In addition, if $\lambda = 1/2$ then, for any given value of $c$ the efficient and equilibrium type of networks also coincide which proves the result. □

It is worth mentioning that this property does not hold when we consider more general anti-coordination games. In particular, if $f \neq e$ there exists a certain range of the parameters for which distribution insensitive states are the least efficient states among the Nash equilibria.

# 5 Further insights

In this paper we have analyzed a setting where agents choose a subset of individuals with whom to play an anti-coordination game, i.e. games where choosing dissimilar actions is individually optimal. In the setup being considered agents interact only if there exists a link between them. The cost of link formation ($c$) is not necessarily distributed as in the classical one- or two-sided models. Instead, we consider a cost-sharing model in which the active agent always supports a higher proportion of the cost (being the partition of the cost specified by the exogenous parameter $\lambda$). We have characterized the Nash equilibria of the game. As $c$ and $\lambda$ change there is a wide variety of Nash architectures: complete, semi-bipartite, bipartite and empty networks. The proportions of agents choosing each action sustainable in equilibrium depend crucially on the values of $c$ and $\lambda$. The effect that an increase of either $c$ or $\lambda$ has over these proportions is similar. Note that, the effect of an increase in $c$ on these proportions coincides with what Bramoullé et al. (2004) showed for the case $\lambda = 1$. Nevertheless, as the model gets closer to a two-sided model the spanning of the equilibrium proportions as $c$ increases is soften. As we increase the value of $\lambda$ (i.e. we make the division of the linking costs more asymmetric) the range of proportions sustainable in equilibrium increases. In particular, when $\lambda$ takes the lowest possible value (i.e. $\frac{1}{2}$) this proportion is uniquely determined in equilibrium.

The main contribution of this paper is that we have presented a general model of network formation that relies on the standard non-cooperative tools but nevertheless allows the implementation of more realistic forms of sharing link costs. More precisely, it studies the effect that different values of the cost share ($\lambda$) has over the results of anti-coordination games played in an endogenous network formation setup. This is a natural extension of a previous work (see Bramoullé et al., 2004) in which the model was strictly one-sided.

## 5.1 Sequential Game

One of the main features of the model is that the acceptance of a proposed link is not modeled explicitly as part of a second stage of the game. Instead, we have assumed that, a passive agent always responds optimally to the proposer's offer. An alternative way of modeling the process of link formation could be to consider a two-stage game. In the first stage, agents can propose links to others and by doing so they incur in a sunk cost equal to $\lambda c$. Then, in the second stage, proposed agents decide whether to accept or not to form these links anticipating that acceptance implies bearing part of the cost (in particular $(1-\lambda)c$). It is straightforward to show that, all Nash equilibria of our primitive model correspond with the equilibrium outcomes of the subgame perfect Nash equilibrium of this alternative sequential version of the game.

## 5.2 Dynamics

Consider a dynamic version of the model in which, from any initial state, with some positive probability each period players receive a revision opportunities over one component of her strategy and assume a myopic best-response adjustment process. The set of limit states of this "unperturbed" dynamics is the set of strict Nash equilibria. To study the robustness of each of these absorbing states, we can use the approach proposed by Kandori, Mailath and Rob (1993), and Young (1993) which assumes that, conditional on receiving a revision opportunity, a player chooses her strategy at random with some small "mutation" probability. The *stochastically stable states* are the states that remain in the support of the invariant distribution as the mutation probability vanishes. Intuitively, this reflects the idea that, even for infinitesimal mutation probability and independent of initial conditions this state materializes a significant fraction of time in the long run. This dynamics has been analyzed for the case $\lambda = 1$ by Bramoullé et al. (2002). It is straightforward to show that the results obtained can be extended to the general model presented here (i.e. where $\lambda \in [\frac{1}{2}, 1]$). Namely, all strict Nash equilibria are stochastically stable and vice versa. Therefore, this dynamics has no selective power in this particular context. Nevertheless, if we consider a variation of this dynamics which assumes that mutations regarding the action are significantly less frequent than mutations regarding links we find that, for certain values of the parameters of the model, the stochastically stable states coincide with the distribution insensitive states. This highlights the importance of distribution insensitiveness in contexts where there is more flexibility in the decision over links than actions.

### 5.3 Coordination games

A straightforward extension of this paper would be to apply this non-cooperative model of network formation to other contexts apart from the anti-coordination games. One possibility would be to consider coordination games. There is already relevant work addressing this issue. For instance, Goyal and Vega-Redondo (2002) study the case $\lambda = 1$ and Meléndez-Jiménez (2002) considers a model where the division of the cost among linked agents is endogenously determined by means of a bargaining process. Both approaches give rise to similar conclusions namely for values of the linking cost below a certain threshold, the complete network with all agents playing the risk dominant equilibrium is selected whereas above this threshold the complete network with all agents choosing the efficient equilibrium is selected. In addition, the work by Jackson and Watts (2001) deals precisely with coordination games in a two-sided model of network formation, however their approach is closer to cooperative game theory. Their results differ from the ones obtained in the previous literature specifically in that, in some cases, the equilibria selected is neither risk dominant nor efficient. If we apply the present model to the coordination games and use stochastic stability to select among equilibria, we find the same qualitative results to the ones mentioned above, where the selected networks are complete graphs and there exists a threshold that determines whether agents are playing the risk-dominant or efficient equilibria.

# 6 Appendix

**Proof of Proposition 2:**

To proof the result we must show the following two statements: (1) If a state has a number of $\beta$-players above the upper bound or below the lower bound, it cannot be sustained in equilibrium for any distribution of active and passive links. (2) For every $n_\beta$ between the lower and upper bounds, there must exist a network structure (i.e., a distribution of active and passive links) with $n_\beta$ players choosing $\beta$ sustained as a strict Nash equilibrium.

We proceed by the successive examination of all the possible domains focusing on the two strict best-responses equations, one for the $\alpha$-player denoted by BR$\alpha$ and one for the $\beta$-players, denoted by BR$\beta$. In general, BR$\beta$ leads to the upper bound , whereas BR$\alpha$ leads to the lower bound. The reason is intuitive: for anti-coordination games the higher the number of people doing one action, the lower the utility of playing that action compared to the benefits of switching to the other action. Therefore, when $\beta$-players are too numerous, BR$\beta$ does not hold. We shall consider separately the domains that induce different types of networks in equilibrium. These are precisely: $\vec{\beta} \rightleftharpoons \vec{\alpha}$, $\vec{\beta} \rightleftharpoons \alpha$, $\beta \rightleftharpoons \vec{\alpha}$, $\beta \rightleftharpoons \alpha$, $\beta \rightarrow \vec{\alpha}$ and $\beta \rightarrow \alpha$. Nevertheless, for most types of networks we have to analyze separately additional cases depending on whether passive links between two agents choosing the same action are profitable or not. For instance, the network $\vec{\beta} \rightleftharpoons \alpha$ has to be analyzed differently depending on whether an $\alpha$-player is willing to pay the cost of maintaining a passive link with another $\alpha$-player or not. The reason is that these considerations are important in order for a $\beta$-player to evaluate her benefits in case of switching to action $\alpha$. Overall, there are 13 different cases that need to be analyzed separately. We will show here the argument for some of these cases since the proof of the remaining ones go along the same lines.

(i) $c < \frac{1}{\lambda} \min\{b, d\}$. Nash networks are complete and essential ($\vec{\beta} \rightleftharpoons \vec{\alpha}$). We shall first compute the upper bound on $n_\beta$. Consider any agent $i$ choosing action $\beta$. Then,

$$\begin{aligned} BR\beta \quad &\Leftrightarrow \quad (n - n_\beta)f + (n_\beta - 1)b - \lambda c(q_i^\alpha + q_i^\beta) - (1-\lambda)c(n - 1 - q_i^\alpha - q_i^\beta) \qquad (10) \\ &> \quad (n - n_\beta)d + (n_\beta - 1)e - \lambda c(q_i^\alpha + q_i^\beta) - (1-\lambda)c(n - 1 - q_i^\alpha - q_i^\beta) \\ &\Leftrightarrow \quad n_\beta < (n-1)\frac{f-d}{f-d+e-b} + 1 \end{aligned}$$

Thus, a $\beta$-player is choosing a best response if and only if $n_\beta < (n-1)p_\beta + 1$. To find the lower bound for $n_\beta$ we have to impose conditions for an $\alpha$-player to be doing a best response. To do this, we will use the expression for the upper bound obtained above and the symmetry of the game. Specifically, we need to exchange the values of $n_\beta$, $f$ and $d$ by $n_\alpha$, $e$ and $b$ in expression (10). Then, substituting $n_\alpha = n - n_\beta$ we obtain the condition,

$$(n-1)p_\beta < n_\beta$$

Notice that, in this case, the existence of a strict Nash equilibrium with a certain number of players choosing each action is independent on the distribution of active and passive links (i.e., the best-response equations are independent of $q_i^\alpha$ and $q_i^\beta$). This implies that statement (2) holds trivially.

(ii) $\frac{1}{\lambda}d < c < min\{\frac{1}{\lambda}b, \frac{1}{1-\lambda}d\}$. Nash networks are of the type $\vec{\beta} \rightleftharpoons \alpha$. Let us first calculate the expression for the upper bound. Consider an agent $i$ choosing action $\beta$. Notice that if she switches to action $\alpha$ she will want to interact with those $\alpha$-players that have proposed a link with her. Therefore,

$$\begin{aligned} BR\beta \quad &\Leftrightarrow \quad (n - n_\beta)f - \lambda c q_i^\alpha - (1-\lambda)c(n - n_\beta - q_i^\alpha) + \\ &\qquad (n_\beta - 1)b - \lambda c q_i^\beta - (1-\lambda)c(n_\beta - 1 - q_i^\beta) \\ &> \quad (n - n_\beta)e - (d - (1-\lambda)c(n - n_\beta - q_i^\alpha)) \\ &\qquad -\lambda c q_i^\beta - (1-\lambda)c(n_\beta - 1 - q_i^\beta) \\ &\Leftrightarrow \quad n_\beta < \frac{n(f-d) + e - b - q_i^\alpha(\lambda c - d)}{f + e - b - d} \end{aligned}$$

We want to find an upper bound for $n_\beta$ thus we assume that agent $i$ is in the best of the possible situations. In other words $q_i^\alpha = 0$. Hence,

$$n_\beta < (n-1)p_\beta + 1$$

To obtain the lower bound consider an agent $j$ choosing action $\alpha$. Then,

$$\begin{aligned} BR\alpha &\Leftrightarrow n_\beta e - \lambda c q_j^\beta - (1-\lambda)c(n_\beta - q_j^\beta) \\ &> n_\beta b - \lambda c q_j^\beta - (1-\lambda)c(n_\beta - q_j^\beta) \\ &\quad +(n - n_\beta - 1)(f - \lambda c) \\ &\Leftrightarrow (n-1)\frac{(f-\lambda c)}{f - \lambda c + e - b} < n_\beta \end{aligned}$$

Thus, we have proved statement (1). In order to show that statement (2) also holds note that, in this case, an $\alpha$-player is choosing a best-response independently of how active and passive links with the $\beta$-players are distributed. Thus, for every $n_\beta$ between the lower and upper bound, we can consider the network structure where links between $\alpha$ and $\beta$-players are always proposed by the $\alpha$-players which we know can be sustained in equilibrium.
(iii) $\frac{1}{1-\lambda}d < c < \frac{1}{\lambda}b$. Nash networks are also of the type $\vec{\beta} \rightleftharpoons \alpha$. Let us first compute the upper bound. Consider an agent $i$ choosing action $\beta$. Notice that, the only difference with the previous case is that, if $i$ switches to action $\alpha$ she will not want to interact with the $\alpha$-players (neither actively nor passively). Therefore,

$$\begin{aligned} BR\beta &\Leftrightarrow (n - n_\beta)f - \lambda c q_i^\alpha - (1-\lambda)c(n - n_\beta - q_i^\alpha) + (n_\beta - 1)b \\ &\quad -\lambda c q_i^\beta - (1-\lambda)c(n_\beta - 1 - q_i^\beta) \\ &> (n - n_\beta)e - \lambda c q_i^\beta - (1-\lambda)c(n_\beta - 1 - q_i^\beta) \end{aligned}$$

$$BR\beta \Leftrightarrow n_\beta < \frac{n(f - (1-\lambda)c) + e - b - q_i^\alpha(\lambda c - (1-\lambda)c)}{f + e - b - (1-\lambda)c}$$

We also assume that $q_i^\alpha = 0$. Then,

$$BR\beta \Leftrightarrow n_\beta < (n-1)\frac{f - (1-\lambda)c}{f + e - b - (1-\lambda)c} + 1$$

To obtain the lower bound consider an agent $j$ choosing action $\alpha$. Note that the best-response equations coincide with the ones obtained in case (ii). Thus,

$$(n-1)\frac{(f-\lambda c)}{f - \lambda c + e - b} < n_\beta$$

As before, the network architecture where the $\alpha$-players propose to the $\beta$-players is sustained in equilibrium whenever $n_\beta$ is between the corresponding lower and upper bounds.
(iv) $max\{\frac{1}{\lambda}d, \frac{1}{\lambda}b\} < c < \min\{\frac{1}{1-\lambda}d, \frac{1}{1-\lambda}b, \frac{1}{\lambda}e\}$. Nash network are of the type $\beta \rightleftharpoons \alpha$. Again, consider an agent $i$ choosing action $\beta$. Then,

$$\begin{aligned} BR\beta &\Leftrightarrow (n - n_\beta)f - \lambda c q_i^\alpha - (1-\lambda)c(n - n_\beta - q_i^\alpha) \qquad (11) \\ &> (n_\beta - 1)(e - \lambda c) + (n - n_\beta - q_i^\alpha)(d - (1-\lambda)c) \\ &\Leftrightarrow n_\beta < \frac{n(f-d) + e - \lambda c - q_i^\alpha(\lambda c - d)}{f + e - d - \lambda c} \end{aligned}$$

As before, we want to find the upper bound for $n_\beta$ in a Nash equilibrium. Thus, we impose $q_i^\alpha = 0$. Then,

$$BR\beta \Leftrightarrow n_\beta < (n-1)\frac{f-d}{f+e-d-\lambda c} + 1$$

To find the lower bound for $n_\beta$ consider $j$ choosing action $\alpha$. Then,

$$\begin{aligned} BR\alpha \quad \Leftrightarrow \quad & n_\beta e - \lambda c q_j^\beta - (1-\lambda)c(n_\beta - q_j^\beta) \\ > \quad & (n - n_\beta - 1)(f - \lambda c) + (n_\beta - q_j^\beta)(b - (1-\lambda)c) \\ \Leftrightarrow \quad & \frac{(n-1)(f-\lambda c) + q_j^\beta(\lambda c - b)}{f - \lambda c + e - b} < n_\beta \end{aligned} \tag{12}$$

Thus to obtain the lower bound, we impose $q_j^\beta = 0$. Then,

$$BR\alpha \Leftrightarrow (n-1)\frac{f-\lambda c}{f-\lambda c+e-b} < n_\beta$$

Nevertheless, up to now we have simply shown statement (1). In order to show statement (2), we will proceed as follows. Let us first find under which conditions over the size of $n_\beta$, an $\alpha$-player is choosing a best-response even in the case that she proposes all the links with the $\beta$-players. To find this condition, simply substitute $q_j^\beta = n_\beta$ in equation (12). Then,

$$(n-1)\frac{(f-\lambda c)}{f+e-2\lambda c} < n_\beta$$

Therefore if the following holds

$$(n-1)\frac{(f-\lambda c)}{f+e-2\lambda c} < n_\beta < (n-1)\frac{f-d}{f+e-d-\lambda c} + 1 \tag{13}$$

a network structure where all links across $\alpha$ and $\beta$-players are proposed by the $\alpha$-players is sustained in equilibrium.

Let us now consider the counterpart condition. That is, under which conditions over the size of $n_\beta$, a $\beta$-player is choosing a best-response even in the case that she proposes all the links with the $\alpha$-players. To find this condition, simply substitute $q_i^\alpha = n - n_\beta$ in equation (11). Then,

$$n_\beta < (n-1)\frac{(f-\lambda c)}{f+e-2\lambda c} + 1$$

Therefore if the following holds

$$(n-1)\frac{f-\lambda c}{f-\lambda c+e-b} < n_\beta < (n-1)\frac{(f-\lambda c)}{f+e-2\lambda c} + 1 \tag{14}$$

a network structure where all links across $\alpha$ and $\beta$-players are proposed by the $\beta$-players is sustained in equilibrium. To conclude, observe that the lower bound provided by equation (13) is lower than the upper bound in equation (14). This implies that, for any value of $n_\beta$ between the bounds provided by the Proposition there exists a network structure that can be sustained in equilibrium. In particular, depending on the value of $n_\beta$, we can consider a network structure such that either all links are proposed by the $\alpha$-players or by the contrary, all links are proposed by the $\beta$-players.

We note that the analysis of the remaining cases uses arguments similar to the ones above and therefore these proofs are omitted.

**Proof of Proposition 3:**

This proposition is obtained by rewriting the intervals where the functions $\psi_\lambda(c)$ and $\varphi_\lambda(c)$ are defined as expressions where the independent variable is $\lambda$ whereas $c$ is a parameter. To illustrate, consider the lower bound $\psi_\lambda(c)$. If $c \leq \min\{\frac{1}{\lambda}d, \frac{1}{1-\lambda}b\}$ then $\psi_\lambda(c) = p_\beta$. For what values of $\lambda$ do we obtain $\psi_c(\lambda) = p_\beta$ ? We can answer this question by simply solving for $\lambda$ in the inequality $c \leq \min\{\frac{1}{\lambda}d, \frac{1}{1-\lambda}b\}$. Notice that, we obtain $max\{\frac{1}{2}, 1-\frac{1}{c}b\} \leq \lambda \leq \min\{\frac{1}{c}d, 1\}$. The same could be done for the remaining cases. □

**Proof of Proposition 6:**

(i) $c < (1/\lambda)min\{b, d\}$. The Nash networks obtained are complete and essential (i.e. $\vec{\beta} \rightleftharpoons \vec{\alpha}$). Consider any agent $i \in N_\alpha$ that supports $q_i^\alpha$ active links with other $\alpha$-players and has $q_i^\beta$ active links with $\beta$-players. Then, in order for this player to be choosing a best response, a necessary and sufficient condition is that,

$$\begin{aligned} BR\alpha &\Leftrightarrow n_\beta e + (n - n_\beta - 1)d - \lambda c(q_i^\alpha + q_i^\beta) - (1-\lambda)c(n - 1 - q_i^\alpha - q_i^\beta) \\ &> n_\beta b + (n - n_\beta - 1)f - \lambda c(q_i^\alpha + q_i^\beta) - (1-\lambda)c(n - 1 - q_i^\alpha - q_i^\beta) \\ &\Leftrightarrow n_\beta > (n-1)\frac{f-d}{f-d+e-b} \end{aligned}$$

We focus now on the counterpart condition for any agent $j \in N_\beta$. Then, in order for this player to be choosing a best response, a necessary and sufficient condition is that,

$$\begin{aligned} BR\beta &\Leftrightarrow (n - n_\beta)f + (n_\beta - 1)b - \lambda c(q_j^\alpha + q_j^\beta) - (1-\lambda)c(n - 1 - q_j^\alpha - q_j^\beta) \\ &> (n - n_\beta)d + (n_\beta - 1)e - \lambda c(q_j^\alpha + q_j^\beta) - (1-\lambda)c(n - 1 - q_j^\alpha - q_j^\beta) \\ &\Leftrightarrow n_\beta < (n-1)\frac{f-d}{f-d+e-b} + 1 \end{aligned}$$

Combining the expressions obtained for $BR\alpha$ and $BR\beta$, the desired conclusion follows.

(ii) $(1/\lambda)b < c < (1/\lambda)\min\{d, e\}$. Nash networks are semi-bipartite graphs of the type $\beta \rightleftharpoons \vec{\alpha}$. Consider any agent $i \in N_\alpha$. Then,

$$\begin{aligned} BR\alpha &\Leftrightarrow n_\beta e + (n - n_\beta - 1)d - \lambda c(q_i^\alpha + q_i^\beta) - (1-\lambda)c(n - 1 - q_i^\alpha - q_i^\beta) \\ &> (n - n_\beta - 1)f - \lambda c q_i^\alpha - (1-\lambda)c(n - n_\beta - 1 - q_i^\alpha) + R(c) \end{aligned}$$

where $R(c) = (n_\beta - q_i^\beta)(b - (1-\lambda)c)$ if $c < \frac{1}{1-\lambda}b$ and $R(c) = 0$ if $c > \frac{1}{1-\lambda}b$. Due to the fact that we want to calculate conditions for the existence of distribution insensitive states we will consider the worst possible situation for an $\alpha$-player. That is, we assume $q_i^\beta = n_\beta$. Under this assumption $R(c)$ is constant and equal to 0. Hence,

$$BR\alpha \Leftrightarrow n_\beta > (n-1)\frac{f-d}{f-d+e-\lambda c}$$

Consider now any agent $j \in N_\beta$. Then,

$$\begin{aligned} BR\beta \quad &\Leftrightarrow \quad (n-n_\beta)f - \lambda c q_j^\alpha - (1-\lambda)c(n-n_\beta-q_j^\alpha) \\ &> \quad (n-n_\beta)d - \lambda c q_j^\alpha - (1-\lambda)c(n-n_\beta-q_j^\alpha) + (n_\beta-1)(e-\lambda c) \\ &\Leftrightarrow \quad n_\beta < (n-1)\frac{f-d}{f-d+e-\lambda c} + 1 \end{aligned}$$

(iii) $(1/\lambda)d < c < (1/\lambda)b$. Nash networks are also semi-bipartite graphs of the type $\overrightarrow{\beta} \rightleftharpoons \alpha$. This case is symmetric to the previous one. Thus, we can simply exchange $d$, $f$ and $n_\beta$ by $b$, $e$ and $n_\alpha$. We obtain,

$$(n-1)\frac{e-b}{e-b+f-\lambda c} < n_\alpha < (n-1)\frac{e-b}{e-b+f-\lambda c} + 1$$

Given that $n_\beta = n - n_\alpha$ we have that,

$$(n-1)\frac{f-\lambda c}{f-\lambda c+e-b} < n_\beta < (n-1)\frac{f-\lambda c}{f-\lambda c+e-b} + 1$$

(iv) $(1/\lambda)\max\{b,d\} < c < (1/\lambda)e$. This case is already tackled in the body of the paper.
(v) $\frac{1}{\lambda}e < c$. Nash networks are of the types $\beta \to \alpha$ and $\beta \to \overrightarrow{\alpha}$. In the first case, links between $\beta$ and $\alpha$-players are only profitable if they are proposed by $\beta$-players. Thus, the are no bidirectional links. In the second case, links between the $\alpha$-players are bidirectional. However, it is straightforward to show that all Nash equilibria are robust to changes in the directions of links formed by two $\alpha$-players. □

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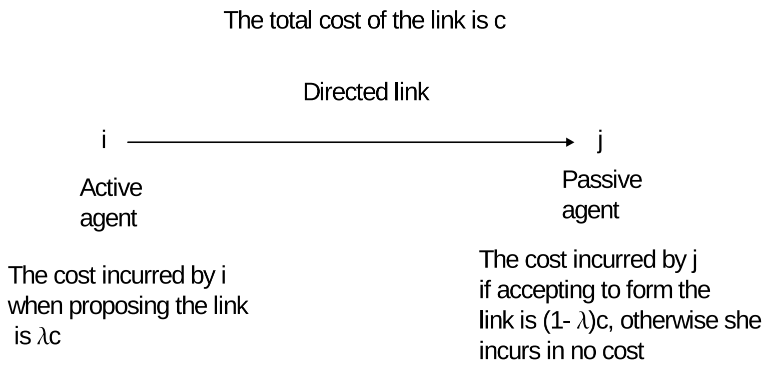


Figure 1: Link formation process.

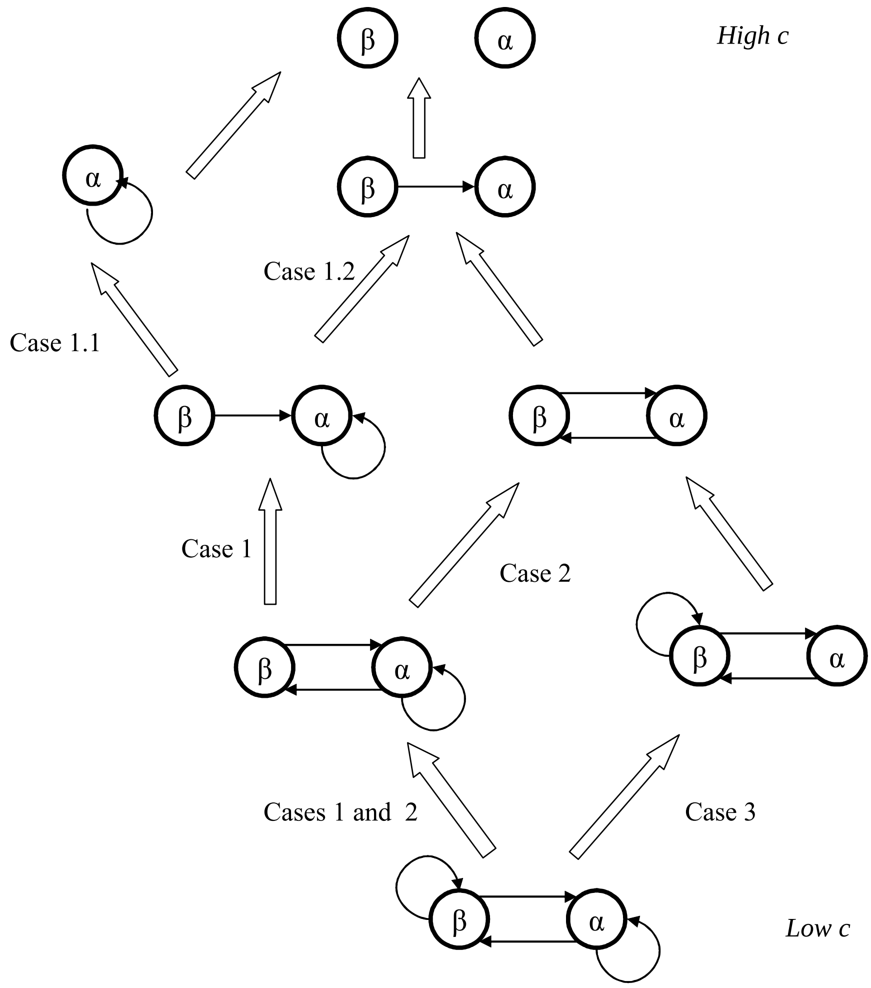


Figure 2: Types of Nash networks found as $c$ increases.

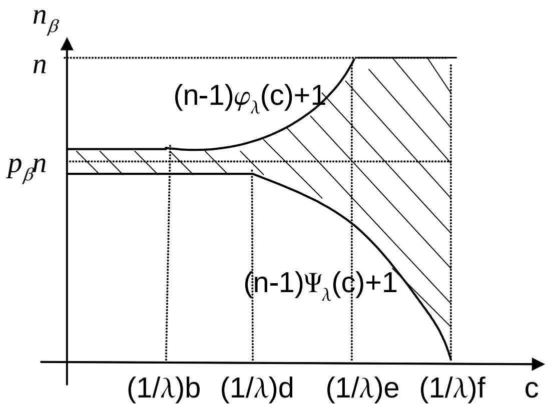


Figure 3: Number of $\beta$-players in equilibrium for Case 2 and $\lambda > \frac{f}{f+b}$.

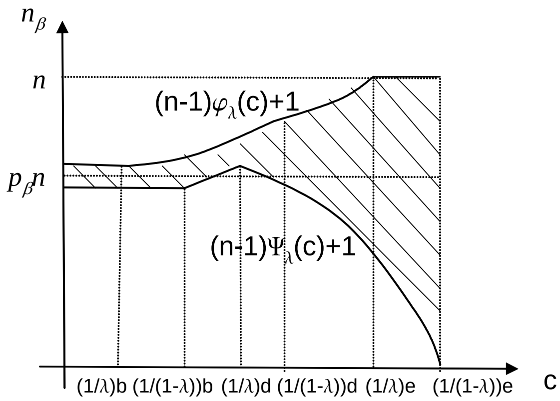


Figure 4: Number of $\beta$-players in equilibrium for Case 2 and intermediate values of $\lambda$. In particular $\lambda < \min\{\frac{f}{f+e}, \frac{e}{e+d}, \frac{d}{d+b}\}$.

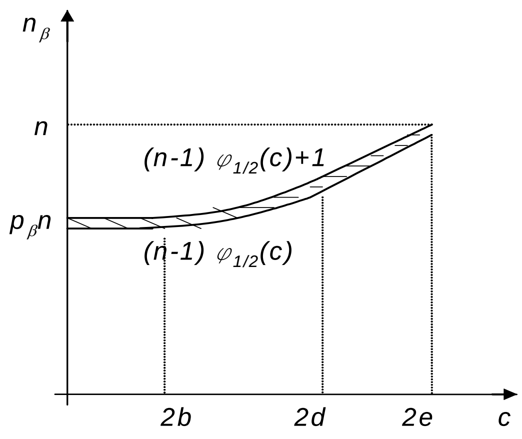


Figure 5: Number of $\beta$-players in equilibrium for Case 2 and $\lambda = 1/2$.

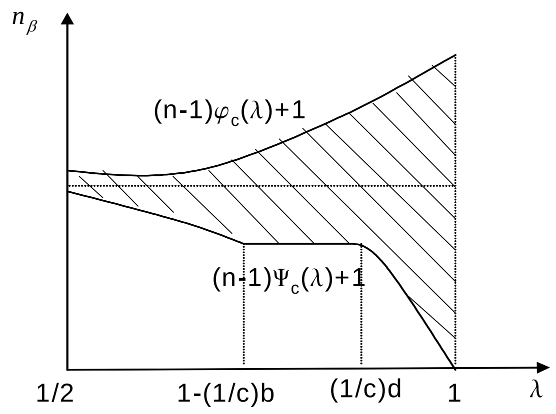


Figure 6: Number of $\beta$-players in equilibrium for $d, 2b < c < e, f\ , b + d$.

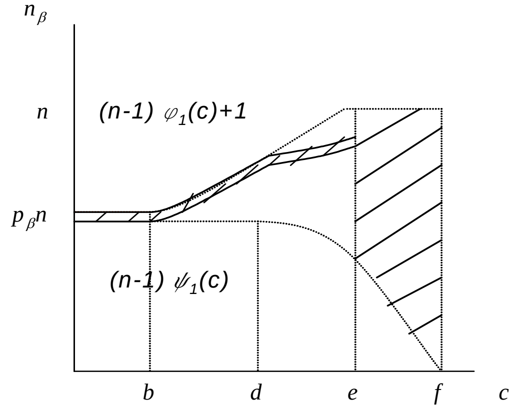


Figure 7: Number of $\beta$-players in distribution insensitive states for Case 2 and $\lambda = 1$.