Ascending Auctions for Gradually Expiring Items *

Ron Lavi†

SISL/IST
California Institute of Technology
ronlavi@ist.caltech.edu

Noam Nisan‡

School of Computer Science and Engineering
The Hebrew University, Jerusalem, Israel
noam@cs.huji.ac.il

Abstract

We consider auction mechanisms for the allocation of M items that are identical to each other except for the fact that the items have different expiration times, and each item must be allocated before it expires. This model seems applicable in many economic situations where both items and buyers have a finite "life-time", e.g. the allocation of transportation tickets. We are interested in situations where players act strategically and may mis-report their private parameters. We first design two auction-like mechanisms and prove that an approximatly efficient allocation is obtained for a wide class of "semi-myopic" selfish behaviors of the players. We then provide a game-theoretic rational justification for acting in such a semi-myopic way. We show that ex-post implementation can not be used in this case, since any such equilibria can not obtain even an approximately efficient outcome. Instead we suggest a new notion of "Set-Nash" equilibria, where we can not pin-point a single best-response strategy, but rather only a set of possible best-response strategies. These strategies are all semi-myopic and thus our auction mechanisms will perform well on any of them. We believe that this notion is of independent interest.

1 Introduction

Mechanism design in an incomplete information setting generally requires strong distributional assumptions. Namely, it is commonly assumed that the correct distribution over the states of the world is known, and, furthermore, that this correct distribution is common knowledge among all participants. The attempt to remove part of these distributional assumptions usually fails, as it leads to an ex-post implementation, which is associated mainly with impossibilities[1].

In this paper we explore, for a representative problem, the other extreme: what types of mechanisms can we design when *no* distributional assumptions are made. The problem we study is the allocation of M items that are all identical except that they "expire" at different times: the first item expires at time 1, the second at time 2, and so on. Players arrive over time, and items must

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[1] Positive results with ex-post equilibria "usually" exist only for single-dimensional domains: in private value settings ex-post implementation is equivalent to dominant strategies implementation, and for interdependent-values settings, see e.g. the impossibility result of Meyer-ter-Vehn and Moldovanu [13]

be allocated at or before their expiration time. Each player j desires any single item between his arrival time, r_j, and his deadline, d_j, and has a value v_j for receiving the item. All information r_j, d_j, v_j is private to player j, and players act rationally to maximize their utility. This model seems applicable to many scenarios in which items are sequentially allocated as time progresses, where both items and players have a finite "life-time".

Our goal in this paper is to design mechanisms that lead to efficient allocations, i.e. to allocate the items so that the sum of values of players that receive an item is maximized. However, as we assume no knowledge of the underlying distribution, we will "accept" mechanisms that lead to approximately optimal outcomes. Specifically, a mechanism is a c-approximation ($c \geq 1$) of the social welfare if it always results in a social welfare of at least the optimal social welfare over c (the parameter c is termed the approximation ratio). However, this relaxation alone cannot compensate for the lack of distributional knowledge. Our first result shows that good approximation mechanisms in dominant strategies (and thus also in ex-post Nash equilibrium) are impossible: every such mechanism cannot have an approximation ratio better than M, the number of items.

At this point, instead of introducing strong distributional assumptions that will enable the emergence of a Bayesian-Nash analysis, we will design "detail free" mechanisms in the spirit of Wilson's critique [17]. The trade-off will be a relaxation of the notion of equilibria: instead of specifying a single tuple of strategies (the equilibrium point) that provides a good approximation ratio, we will specify a large set of strategies with the property that the mechanism will perform well on *any* of them. Our strategic analysis will not be able to pinpoint exactly which of these strategies will be rationally chosen, but will rather suggest only that *one* of them will be chosen – this is provably enough to guarantee good performance.

To show this, we give a new analysis of two variants of classic ascending auctions, under a wide class of possible player behaviors. The first auction we consider is a natural adaptation of the iterative auction of Demange, Gale, and Sotomayor [5]. *The Online Iterative Auction* constantly maintains a current price p_t and a current winner win_t for every item t. At each time t, each player (in his turn) may place his name as the temporary winner of some item t' (bid on t'), deleting the previous temporary winner, and increasing the price by some fixed small δ (a player can be a temporary winner only for one item). When none of the players wishes to bid, the time t phase ends: item t is sold to player win_t for a price of $p_t - \delta$. At time $t+1$ the prices and temporary winners from time t are kept, and the auction continues similarly.

In the *offline* setting, where all players arrive at time 1, Demange et al. [5] show that if all players behave *myopically*, i.e. always bid on the item with the lowest price among those that interest them, then the auction will reach the optimal allocation. Moreover, such myopic behavior is an ex-post Nash equilibrium (this was later shown by Gul and Stacchetti [9]). But what will the player choose, facing this auction in the online setting (when players arrive over time)? This depends on the beliefs of the player about the future: if he fears that new competitive bidders will arrive in the future, he may bid aggressively for earlier items, offering a higher price for them but reducing his risk of future competition. To incorporate such considerations, we call a player *semi-myopic* if he always bids on some item with price lower than his value (not necessarily the item with the lowest price, as the myopic behavior requires). Thus there exist many semi-myopic behaviors, that represent different beliefs. We show that the auction will obtain near optimal allocation under *any* combination of such behaviors: If all players are semi-myopic then the Online Iterative Auction achieves a 3-approximation of the welfare.

The second auction we consider is *The Sequential Japanese Auction*: Item t is sold at time t

using a classic one-item ascending auction with slight modifications. Surprisingly, we show that this auction has a similar structure to the previous one (in our setting). We define a myopic behavior that leads to the optimal allocation in the offline case (when all players arrive at time 1), and, similarly to above, a family of semi-myopic behaviors aimed to capture players' uncertainties about the future. For this auction we again are able to show that a 3-approximation is obtained for *every* combination of semi-myopic behaviors.

But why should the players play as we expect? We now turn to give a more accurate game-theoretic analysis of the players' behaviors. As approximately optimal auctions with ex-post equilibrium do not exist, we instead seek an equilibrium notion that will capture the idea advocated above, i.e. that the best we can do is recommend on a set of strategies, and not on a specific, single strategy. Our notion of "set equilibria" captures the situation where we describe a subset $R_i \subseteq S_i$ of "recommended strategies" (best response strategies) to choose from, for each player i, instead of describing a single strategy r_i as the equilibrium point. Formally, we say that the strategy sets R_i are in "Set-Nash equilibrium" if for any player i, and any strategy combination of the other players $s_{-i} \in R_{-i}$, player i has a best response to s_{-i} in R_i. Thus we require that a best response to any tuple of recommended strategies of the others may be found within the recommended strategies of player i. This becomes equivalent to regular Nash equilibrium when $|R_i| = 1$ for all i. It should be pointed out that there always exists a trivial Set-Nash equilibrium in which the recommended strategies are the entire set of strategies. Therefore this notion is interesting only when one can guarantee some performance bound whenever players play any one of their recommended strategies, as we do.

We also provide some discussion on ways to strengthen the basic definition. We describe a hierarchy of four "set equilibria" notions, with a growing strength. While, for our motivating example, we were able to use only the basic definition, we believe that the complete hierarchy will turn out to be useful for other models, in which ex-post implementation is impossible, and one wishes to construct detail-free mechanisms, and to avoid making distributional assumptions.

Returning to our model, we show that both our online ascending auctions have Set-Nash equilibria that are all semi-myopic. We leave the description of the appropriate sets of recommended strategies to the body of the paper. The main point we arrive at is that players do not have a clear incentive to deviate outside of these sets of recommended strategies; and when they do stay inside the set of recommended strategies, the mechanism obtains a 3-approximation.

The rest of this paper is organized as follows. The model and basic definitions are given in section 2. In section 3 we describe the two online ascending auctions, and analyze their performance under semi-myopic behaviors. Section 4 returns to the strategic setting, showing that no dominant strategies implementation can achieve a good approximation ratio. In section 5 we define the notion of a Set-Nash equilibrium and a hierarchy of three stronger notions. We discuss their properties and the relevant literature. In section 6 we analyze our auctions accordingly. Appendix A describes some useful observations about the offline allocation problem, used as basic building blocks in our proofs.

2 Model and Basic Definitions

Items: We wish to sell M identical items with different expiration times. W.l.o.g. we assume that the first item expires at time 1, the second at time 2, and so on. Each item must be sold (and received by the buyer) at or before its expiration time.

Players: The potential buyers of the items (players/bidders) arrive over time. Player i arrives to the market at time $r(i)$, and stays in the market for some fixed period of time, until his deadline $d(i)$. We assume w.l.o.g. that the arrival and departure times are integers[2]. Each player desires only one item (unit demand), that expires no earlier than his arrival time. He must receive it at or before his departure time[3]. Player i obtains a value of $v(i)$ from receiving such an item, otherwise his value is 0. We assume w.l.o.g. that different players have different values[4].

We assume the standard game-theoretic setting: Player i privately obtains his variables $r(i)$, $d(i)$, and $v(i)$, at time $r(i)$. He acts selfishly in order to maximize his own utility: his obtained value minus his price. I.e., a player may arrive at or after his true arrival time, and declare or act as if he has any value, and any deadline.

We defer questions about the exact knowledge of the players, besides their own private parameters, until section 5 below, where we analyze the strategic behavior.

Our goal: We aim to design allocation mechanisms that will prdocuce efficient outcomes, i.e. that will maximize the social welfare: the sum of (true) values of players that receive an item. In the design and the analysis of our mechanisms, we will not assume knowledge of the underlying distribution. Instead, we will require the mechanism to perform approximately well even in the *worst case*:

Definition 1 (A c-approximation allocation rule) *An allocation rule is a c-approximation ($c \geq 1$) of the social welfare if it always produces an outcome with social welfare of at least the optimal social welfare over c. The parameter c is termed the approximation ratio.*

By employing worst case arguments and completely avoiding any distributional assumptions, we gain the ability to construct detail-free mechanisms. I.e., our mechanisms will not be tailored for one specific type of distribution (e.g. a poisson arrival process), but will perform provably well for all cases.

A valid critique, however, on such an analysis, is that it can be viewed as switching from one extreme to the other: instead of assuming full and correct knowledge of the underlying distribution, we avoid any such assumptions all together. Indeed, it seems that the "right" set of assumptions should be somewhere in the middle – constructing a detail free mechanism that will perform well assuming a large family of distributions. But for advancing towards this purpose, it seems extremely important to completely understand the worst-case extreme as well.

Basic notations: Player i is *active* at time t if $r(i) \leq t \leq d(i)$, and i did not win any item before time t. Let A_t be the set of all active players at time t. An *allocation* is a mapping of items to players such that, if player i receives item t, then $r(i) \leq t \leq d(i)$. Let X_t be an allocation of items $t, ..., M$. $X_t[d]$ denotes the player that receives item d according to X_t, and $X_t[d_1, d_2] = \cup_{d=d_1}^{d_2} X_t[d]$, the set of players that receive items d_1 through d_2. By a slight abuse of notation we also use X_t as the set of players $X_t[t, M]$. The *value* of X_t is $v(X_t) = \sum_{d=t}^{M} v(X_t[d])$, i.e. the welfare obtained by X_t. A set S of players is *independent* with respect to items $t, ..., M$ if there exists an allocation of (part of) the items $t, ..., M$ such that every player in S receives an item.

[2]as actions in a non-integral time point can be deferred to the next integral point with no affect.

[3]Our auctions also fit the more severe restriction that player i cannot get an item $t > d(i)$. E.g., player i cannot attend Saturday's show if he is leaving on Friday, even if he receives the ticket before Friday.

[4]I.e. fix some arbitrary order over players, and set $v(i) \succ v(j)$ iff $v(i) > v(j)$ or $v(i) = v(j)$ and $i \succ j$.

The offline allocation problem: The offline problem, in which all players arrive at time 1, is a matroid: a set of *players* is independent if there exists an allocation of (part of the items) to these players. This is known [10] for the unit-demand scheduling problem, which is equivalent to ours. This matroid structure is used extensively in our proofs. See Appendix A for more details.

3 Two Online Ascending Auctions

We first describe online adaptations of two well-known ascending auctions. These have the property that players do not have to choose specific actions for the auction to perform well: a 3-approximation is obtained for a large, reasonable family of behaviors that we term "semi-myopic". Under any such player behaviors, each of our auctions belongs to a general family of semi-myopic allocation rules, that we characterize. We then show that any semi-myopic alllocation rule obtains a 3-approximation, and therefore conclude that our auctions lead to a near optimal allocation for any choice of semi myopic behaviors of the players.

In this section, we focus on the quality of allocations that the auctions achieve. Therefore we give only intuitive justifications for the player behaviors that we assume. For the same reason, we also omit few technicalities about prices and tie-breaking rules from the definitions. All these are detailed when we analyze the strategic properties of our auctions, below.

3.1 The Online Iterative Auction

We consider an online adaptation of the iterative auction of Demange, Gale, and Sotomayor [5]:

Definition 2 (The Online Iterative Auction (intuitive version)) *The Online Iterative Auction constantly maintains a current price p_t and a current winner win_t for every item t. These are initialized to zero at $t = 0$, and updated according to players' actions at each time t, as follows:*

- *Each player, in his turn, may place his name as the temporary winner of some item t', causing the previous winner to be deleted, and the price to increase by some fixed small δ. A player cannot perform this action, and must relinquish his turn, if he is already a temporary winner.*
- *When none of the players that are not temporary winners wishes to place their names somewhere, the time t phase ends: item t is sold to the player win_t for a price of $p_t - \delta$.*
- *At time $t + 1$ the prices and temporary winners from time t are kept. If additional players arrive then the auction continues according to the above rules.*

Before analyzing the online auction, it is useful to take a glimpse at the offline case, in which all players arrive at time 1. This is a special case of the unit-demand model studied by [5], [9]:

Definition 3 ([5]) *Player i has a* **myopic strategy** *in the iterative auction if, in his turn, he always places his name on the item $t \leq d(i)$ with the minimal price, unless the minimal price $\geq v(i)$, in which case he does not place his name at all.*

Lemma 1 ([5], [9]) *If all players are myopic and arrive at time 1 then the online iterative auction obtains the optimal allocation. Furthermore, if all other players are myopic then player i will maximize his utility by playing myopically.*

In the online setting, however, a player might not be completely myopic, depending on his beliefs about the future. For example, he may bid aggressively for the current item, not placing his name on future items at all. This is reasonable if he anticipates tight competition from players that will arrive later on. Viewing this behavior as one extreme, and the completely myopic behavior as the other, it seems that any combination of the two cannot be "ruled-out". On the other hand, a player might choose not to participate at all for some time units – if, for example, there are M high valued players that desire any item 1 through M, but they all do not participate up to time M, then the resulting welfare will be low. As it turns out, this is the only type of behavior we need to exclude:

Definition 4 *Player i is* **semi-myopic** *if, in his turn, i places his name on some item t with $p(t) \le v(i)$ and $r(i) \le t \le d(i)$ (not necessarily the one with the lowest price). If there is no such item, i stops participating.*

Theorem 1 *If all players are semi-myopic then the online iterative auction achieves almost a 3-approximation: $v(OPT) \le 3 \cdot v(ON) + 2 \cdot M \cdot \delta$, where OPT, ON are the optimal, online allocations.*

The proof is given in section 3.3 below, where we show that, under any semi-myopic behavior, the online iterative auction follows a semi myopic allocation rule, hence obtains the desired approximation.

3.2 The Sequential Japanese Auction

A different possibility is to sell item t at time t using a simple one item ascending auction:

Definition 5 (A Japanese Auction) *The (classic, one item) Japanese auction operates as follows: An auctioneer gradually raises a price, starting from 0. Each participating player should decide whether to drop out or to stay (once a player drops out, he cannot join again), as the price ascends. The price stops increasing exactly when all players, besides one, have dropped out. The winner is the player that did not drop out, and he pays the price that was reached.*

A natural adaptation of this to the online case is:

Definition 6 (The Sequential Japanese Auction (intuitive version)) *The Sequential Japanese Auction sells each item t at time t, separately, using a Japanese auction with one modification: the participants are allowed to observe how many drop-outs occur as the price ascends (and to incorporate this into their drop-out decision).* [5]

As before, it is useful to first consider this auction for the offline case, in which a rather surprising notion of myopic behavior leads to the optimal allocation:

Definition 7 *Player i is* **myopic** *in the Sequential Japanese Auction if, in the auction of any time t, (for $r(i) \le t \le d(i)$), he drops exactly when either the price reaches $v(i)$, or when there are exactly $d(i) - t$ other players that did not drop yet.*

[5] Prices are also modified. The time-t-winner pays the highest price among all time-t'-auctions in which he tied the time-t'-winner. Defining "a tie" is delicate, and requires the players to drop simultaneously. See section 6.4.

The logic for dropping when $d(i) - t$ players remain is that at this point the player is assured that there are enough items before his deadline to be allocated to all bidders who are willing to pay the current price.

Lemma 2 *If all players are myopic and arrive at time* 1 *then the Sequential Japanese Auction obtains the optimal allocation.*

Our assumption that player have different values is important here. It is not hard to verify that this lemma is actually a special case of theorem 6 from the online strategic setting (specifically, it follows from Lemma 8). In this case, a myopic behavior (in the offline case) is a best response when all others are myopic only when using the modified prices of section 6.4.

In the online setting, again, players might not play myopically, and may insist on closer items (i.e. stay longer in the auction) if they anticipate much competition in the future. All we wish is that players will not drop out "too soon". Indeed, dropping out early in the auction also have disadvantages, as future auctions might be much more competitive, due to new arriving players.

Definition 8 *Player i's strategy is* **semi-myopic** *(for the Sequential Japanese Auction) if, at every time t, he drops no later than when the price reaches his value, $v(i)$, and no earlier than when only $d(i) - t$ other players remain in the auction.*

Theorem 2 *If all players play semi-myopic strategies then the Sequential Japanese Auction achieves a* 3*-approximation.*

In a similar manner to the iterative auction above, this theorem is proved by showing that, under any semi-myopic behavior, the Sequential Japanese Auction results in a semi myopic allocation rule. The proof is given is section 3.3 below.

3.3 Semi-Myopic Allocation Rules

For each combination of player strategies, the above auctions are associated with a different allocation rule. In order to analyze their performance for a family of strategies, we therefore need to characterize a family of allocation rules, that we call semi-myopic allocation rules. The main point is that *any* semi myopic allocation rule obtains a 3-approximation of the welfare.

Specifically, the *current best schedule* at time t, S_t, is the allocation with maximal value among all allocations of items $t, ..., M$ to the active players, A_t[6]. Define

$$f_t = \{ \ j \in S_t \mid S_t \setminus j \text{ is independent w.r.t items } t+1, ..., M \ \}, \tag{1}$$

The set f_t contains all players that can receive item t, when one plans to allocate items $t, ..., M$ to the players of S_t (i.e. these are all the potentially *first* players). Now define the critical value at time t, v_t^*, as:

$$v_t^* = \begin{cases} 0 & S_t \text{ is independent w.r.t. items } t+1, ..., M \\ \min_{j \in f_t}\{v(j)\} & \text{otherwise} \end{cases}$$

All active players with value larger than v_t^* must belong to S_t, because of its optimality (w.l.o.g the first player in S_t has value v_t^*, and if there was a higher valued player outside of S_t, we could switch

[6] There exists one such allocation, by the matroid structure, and since different players have different values.

between them and increase the value of S_t). Thus, it seems reasonable not to allocate item t to a player with value less than v_t^*, as this player cannot belong to any optimal allocation. Surprisingly, this condition is enough to obtain approximately optimal allocations:

Definition 9 (A semi myopic allocation rule) *An allocation rule is semi myopic if every item t is sold at time t to some player j with $v(j) \geq v_t^*$.* [7]

Lemma 3 *The Online Iterative Auction with semi-myopic players and the Sequential Japanese Auction with semi-myopic players are both semi myopic allocation rules*[8].

proof: We first show the claim for the Online Iterative Auction. If $v_t^* = 0$ then, trivially, $v(win_t) \geq v_t^* - \delta$. Thus assume that $v_t^* > 0$. Let Y_t be the allocation of items to the temporary winning players at the end of time t iterations. According to claim 29 in section A.1, f_t is independent w.r.t items $t+1, ..., M$ if and only if $v_t^* = 0$. Therefore f_t is not independent, so there exists some player $j \in f_t$ such that $j \notin Y_t[t+1, M]$. Since $j \in f_t$ then $v(j) \geq v_t^*$. if $j = Y_t[t]$ $(= win_t)$ then we are done. Otherwise, j is not a temporary winner at the end of time t iterations. Since j is semi-myopic, this implies that $v_t^* \leq v(j) < p(t)$. Let $i = win_t$. Since i is also semi-myopic then $v(i) \geq p(t) - \delta$. Therefore $v(win_t) \geq v_t^* - \delta$, as needed. This concludes the claim for the Online Iterative Auction.

For the Sequential Japanese Auction, we show that the winner has value at least v_t^*. Let $j \in f_t$ be the first player in f_t that dropped. If he dropped because the price reached v_j then the winner has value at least v_j, which is at least v_t^*. Otherwise there were at most $d(j) - t + 1$ players that did not drop yet, including j. By claim 24[9], $d(j) - t + 1 \leq |f_t|$. Since no player in f_t dropped yet, it follows that every player that did not drop yet belongs to f_t, hence the winner belongs to f_t and has value at least v_t^* by definition. ∎

The family of semi myopic allocation rules can be viewed as the entire range between the following two extremes: the first is the greedy allocation rule, that always chooses the player with maximal value[10], and the second is the "myopic" allocation rule that always chooses the player that determined v_t^*. These two extremes are 2-approximations (both were studied in the context of online scheduling [11, 2]). The entire family has only a slightly larger approximation ratio:

Lemma 4 *Any semi myopic allocation rule is a 3-approximation of the welfare (and this is tight).*

Corollary 1 *The Online Iterative Auction and The Sequential Japanese Auction are both a 3-approximation of the welfare.* [11]

Proof of Lemma 4: We will show that any allocation rule that produces an allocation ON has $v(OPT \setminus ON) \leq 2\sum_{t=1}^{M} v_t^*$, where OPT is the optimal allocation. From this, the lemma follows immediately, as any semi-myopic allocation rule has $v(ON[t]) \geq v_t^*$, and therefore $v(OPT) = v(OPT \setminus ON) + V(ON) \leq 2\sum_{t=1}^{M} v_t^* + v(ON) \leq 2 \cdot v(ON) + v(ON) = 3 \cdot v(ON)$. We first prove two useful claims:

[7] A worst-case approximation cannot sell item t before time t, as a player with high value only for t may appear.

[8] For the online iterative auction, we show that $v(win_t) \geq v_t^* - \delta$, which will later imply that it is "almost' a 3-approximation: $v(OPT) \leq 3 \cdot v(ON) + 2 \cdot M \cdot \delta$

[9] We can assume that there are no ϵ players in f_t, otherwise $v_t^* = 0$ and the claim trivially holds.

[10] Interestingly, this is a special case of the greedy algorithm of [12] for combinatorial auctions with sub-modular valuations. They study the offline case, but it is easy to verify that their algorithm actually works online.

[11] To be completely precise, the Online Iterative Auction is "almost" a 3-approximation: for any scenario, $v(OPT) \leq 3 \cdot v(ON) + 2 \cdot M \cdot \delta$, where OPT, ON, is the optimal and semi-myopic allocations, respectively, and δ is the price increment of the auction.

Claim 1 *Let A, B be sets of players, where $A \subset B$. Let S_A, S_B be the allocation with optimal value for A, B, respectively (both are over the same set of items). Then if $j \in A$ but $j \notin S_A$ then $j \notin S_B$*

proof: Assume by contradiction that there exists $j \in S_B \cap A$ but $j \notin S_A$. Notice that S_A and S_B are both independent sets of the matroid over players in B. Notice also that, by the contradiction assumption, $S_A \nsubseteq S_B$, otherwise also $S_A \cup j \subseteq S_B$, implying that $S_A \cup j$ is independent, with players only from A, contradicting the maximality of S_A. Therefore, since $j \in S_B \setminus S_A$, there exists $j' \in S_A \setminus S_B$ such that $S_A \setminus j' \cup j$ and also $S_B \setminus j \cup j'$ are both independent. From the maximality of S_A and since $j \in A$, the first condition implies that $v(j') > v(j)$. But then we obtain a contradiction to the maximality of S_B. ∎

Claim 2 *Let S be the allocation with maximal value over the set of players A and the set of items $t, ..., M$. Assume that S is not independent w.r.t items $t+1, ..., M$. Let $j \in S$ be the player with minimal value such that $S \setminus j$ is independent w.r.t items $t+1, ..., M$. Then $S \setminus j$ has maximal value among all independent sets w.r.t items $t+1, ..., M$ and players in A.*

proof: Denote $S' = S \setminus j$. Suppose by contradiction that the maximal allocation X over items $t+1, ..., M$ has $v(X) > v(S')$. If $j \notin X$ then this contradicts the maximality of S, as $X \cup j$ is independent w.r.t items $t, ..., M$. Otherwise $j \in X \setminus S'$. $S' \not\subset X$, since otherwise $S = S' \cup j \cup X$ contradicting the fact that S is not independent w.r.t items $t+1, ..., M$. Hence there exists $j' \in S' \setminus X$ such that $X \setminus j \cup j'$ and $S' \setminus j' \cup j$ are independent w.r.t items $t+1, ..., M$. Therefore $S \setminus j'$ is independent w.r.t items $t+1, ..., M$, and from the choice of j it follows that $v(j) < v(j')$, contradicting the maximality of X. ∎

We now prove our main claim: any allocation rule that produces an allocation ON has $v(OPT \setminus ON) \leq 2\sum_{t=1}^{M} v_t^*$. Fix some scenario, and let OPT and ON be the optimal and online allocations for this scenario. We describe $f : OPT \setminus ON \rightarrow \{1, ..., M\}$ such that f is 2 to 1 and $v(j) \leq v_{f(j)}^*$ for any $j \in OPT \setminus ON$. From this, the claim immediately follows.

The function f is defined as follows. Let X_t be the optimal allocation of items $t+1, ..., M$ among players in $OPT[1, t] \setminus ON$. For any $j \in OPT \setminus ON$ (say $j = OPT[t']$), let $t_j^* = min\{t \geq t' \mid j \notin X_t\}$. Then we fix $f(j) = t_j^*$.

Claim 3 *For any $j \in OPT \setminus ON$, $v_{f(j)}^* \geq v(j)$.*

proof: Let $t = f(j)$. First notice that $j \in A_t$: $j \notin ON$, $r(j) \leq t$ as $j \in OPT[1, t]$, and $d(j) \geq t$ since either $j \in X_{t-1}$ or $j = OPT[t]$. Let $m_t \in S_t$ be the player who determined v_t^*, (if $v_t^* = 0$ then set $m_t = null$, so $S_t \setminus m_t = S_t$). We first show that, by claim 1, $j \notin S_t \setminus m_t$: define A as $OPT[1, t] \setminus ON$ minus all players with deadline $< t$, and $B = A_t$. Clearly $A \subseteq B$. By definition, X_t is optimal for A (over items $t+1, ..., M$). $S_t \setminus m_t$ is optimal for B (over items $t+1, ..., M$): if $m_t = null$ this follows from the optimality of S_t, and if $m_t \neq null$ this follows from claim 2. Therefore, since $j \notin X_t$ then $j \notin S_t \setminus m_t$. If $j \neq m_t$ then $j \notin S_t$, and since $j \in A_t$ it follows from the optimality of S_t that $v(j) \leq v(m_t)$. If $j = m_t$ then this trivially holds. Therefore $v(j) \leq v(m_t) = v_{f(j)}^*$, and the claim follows. ∎

Claim 4 *f is 2 to 1.*

proof: Fix any time t. We need to show that f maps at most two players to t. Let $j_1 \in X_{t-1}$ be the player with minimal value such that $X_{t-1} \setminus j_1$ is an allocation of items $t+1, ..., M$, and denote

$Y = X_{t-1} \setminus j_1$ (if X_{t-1} itself is independent w.r.t items $t+1, ..., M$ then set $Y = X_{t-1}$). If $X_t \subseteq Y$ then by the optimality of X_t it follows that $X_t = Y$ and the claim follows: by definition, f maps only j_1 and $OPT[t]$ to t. Otherwise, $X_t \setminus Y \neq \emptyset$. We first show that $X_t \setminus Y = \{OPT[t]\}$. This is implied by claim 1: set $A = OPT[1, t-1] \setminus ON$, and $B = OPT[1, t] \setminus ON$. Since Y is optimal for A (by claim 2) and X_t is optimal for B (by definition) it follows that, if $j \in OPT[1, t-1]$ but $j \notin Y$ then $j \notin X_t$, i.e. that $X_t \setminus Y = \{OPT[t]\}$. To conclude, we observe that X_t is a base in the matroid over items $t+1, ..., M$ and players $OPT[1, t] \setminus ON$, and that Y is an independent set of that matroid. Therefore $|Y \setminus X_t| \leq |X_t \setminus Y| = 1$, and thus $|X_{t-1} \setminus X_t| \leq 2$. Since $OPT[t] \in X_t$ then, by definition, the players mapped to t are exactly those in $|X_{t-1} \setminus X_t|$, and the claim follows. ∎
This concludes the proof of Lemma 4. ∎

The following example shows that the 3-approximation factor analysis is tight:

Example 1 *Consider the following scenario for three items. At time 1 arrive two players, j_1 has value ϵ and deadline 1 and j_2 has value 1 and deadline 2. It is easy to verify that $v_1^* = 0$, and so the online allocation rule allocates item 1 to j_1. At time 2 arrive two additional players, j_3 has deadline 2 and j_4 has deadline 3, and both have a value of 1. Therefore $v_2^* = 1$ and the online allocation rule chooses j_4. At time 3 no new players arrive, so item 3 remains unallocated by the online allocation rule. Therefore its welfare is $1 + \epsilon$. The optimal welfare is, however, 3, as needed.*

4 The Impossibility of Truthful Approximations

We now move from performance considerations to game-theoretic ones, in order to analyze player strategies. Since our goal is to find approximately optimal allocations with respect to the *true* variables of the players, we would prefer to design a truthful mechanism, i.e. an allocation rule with price functions such that, regardless of how the other players act, player i will maximize his utility by declaring his true variables. Let us briefly formally re-state the definition of truthfulness. Let T_i be the domain of all valid player i types/bids $(r(i), v(i), d(i))$, and let $T_{-i} = \times_{j \neq i} T_j$. Consider the allocation constructed by the mechanism upon receiving the type $b_i \in T_i$ from player i and $b_{-i} \in T_{-i}$ from the other players, and let $v(i, b)$ be the value that player i obtains from this allocation, i.e. $v(i)$ if i receives one of his desired items, and 0 otherwise.

Definition 10 (Truthfulness) *A mechanism is truthful if there exist price functions $p_i : T_1 \times \cdots \times T_n \rightarrow \Re$ such that, for any i, any $b_{-i} \in T_{-i}$, any true type $b_i \in T_i$, and any $\tilde{b}_i \neq b_i$[12]:*

$$v(i, b_i, b_{-i}) - p_i(b_i, b_{-i}) \geq v(i, \tilde{b}_i, b_{-i}) - p_i(\tilde{b}_i, b_{-i}).$$

Such a property is highly desirable, as it guarantees that each player will be motivated to reveal his true type, by an argument similar to the traditional worst-case arguments of Computer Science. Indeed, many recent examples show truthful mechanisms for various models. However, for our model, no such mechanism performs well:

Theorem 3 *Any truthful deterministic mechanism for our online allocation problem cannot obtain an approximation ratio better than M.*

[12] We actually restrict the possible $\tilde{b}_i$'s such that $\tilde{r}_i \geq r_i$.

proof: Assume w.l.o.g. that a player that does not win any item pays 0. This implies that i's price must not be higher than his value.

Claim 5 *Fix some truthful deterministic mechanism with some approximation ratio c. Then, for any player i with $r(i) = 1$ there exists a price function $p_i : T_{-i} \rightarrow \Re$ such that, for any combination of players that arrive at time 1, b_{-i}:*

- *If $v(i) > p_i(b_{-i})$ then i wins item 1 and pays $p_i(b_{-i})$ (regardless of his deadline).*
- *If $v(i) < p_i(b_{-i})$ then i does not win any item.*

proof: Fix any combination of players that arrive at time 1, b_{-i}. Suppose first that i has deadline equal to 1. For this case, the player becomes one parameter, and by truthfulness there exist a price function according to the claim [1][13].

We now show that this function p_i satisfies the conditions of the claim, regardless of i's deadline. Fix any deadline $d(i)$ of i. If $v(i) > p_i(b_{-i})$ then i must win some item until his deadline, otherwise he can declare $\tilde{d_i} = 1$ and have strictly better utility. But then, if i does not win item 1, the adversary will produce players with higher and higher values, forcing the mechanism not to allocate any item to i in order to maintain the approximation ratio, thus contradicting truthfulness. Therefore i will receive item 1. He will pay $p_i(b_{-i})$ as otherwise, if he pays a higher price, he will declare $\tilde{d_i} = 1$ and will reduce his price, and if he pays less, then if i will have deadline equals 1 he will declare $d(i)$ instead, thus still winning item 1 but paying less. Therefore the function p_i satisfies the first condition.

Suppose now that $v(i) < p_i(b_{-i})$, and suppose there exists a scenario in which i wins one of his desired items. His price must be at most $v(i) < p_i(b_{-i})$. But then, if i had some value larger than $p_i(b_{-i})$ he would have been better off declaring $v(i)$ instead, by this still winning but paying less. Therefore i cannot win any item at all, and the claim follows. ∎

We can now quickly finish the proof of theorem. Fix any price functions $p_i : T_{-i} \rightarrow \Re$. For any $\epsilon > 0$ we will show that there exist player types $b_1, ..., b_M$ such that, for all i, $r(i) = 1$, $d(i) = M$, $1 \le v(i) \le 1 + \epsilon$, and $v(i) \ne p_i(b_{-i})$. By the above claim, it follows that the mechanism can obtain welfare of at most $1 + \epsilon$, while the optimal allocation is at least M, and the theorem follows. To verify that such types exist, fix $L > M$ real values in $[1, 1 + \epsilon]$. Choose M values $v(i)$ uniformly at random from these L values. Then, for any given i, $Pr(v(i) = p_i(v(-i))) \le 1/L$, as the values were drawn i.i.d. Thus, $Pr(\exists i,\ v(i) = p_i(v(-i))) \le M/L < 1$, hence there exist a choice of values with $v(i) \ne p_i(v(-i))$ for all i. ∎

Remark 1: Although the proof utilizes an extreme scenario with players with very large values, the worst case ratio occurs in common, simple scenarios, as the proof demonstrates. I.e., since the mechanism defends itself against such extremes, it must make wrong decisions even in simple cases.

Remark 2: A simple truthful deterministic M-approximation exists: For any player i, set p_i to be the highest bid received in time slots $1, ..., t$, excluding i's own bid. Sell item t to player i if and only if $v(i) > p_i$, for a price of p_i.

[13] The argument essentially states that, if i wins for some $v(i)$ then he wins with any higher value, and pays the same. Therefore there exists a threshold value $p_i^* = p_i^*(b_{-i})$, such that i wins and pays p_i^* if $v(i) > p_i^*$, and looses otherwise.

5 A Game-Theoretic Framework

Our main motivation at this point is to justify the assumption that players will behave "as expected". We desire a rational justification, i.e. one that shows that expected strategies are, in some sense, utility maximizers for the players. The settings that we are interested in are ones in which "recommended" strategies are indeed to be intuitively expected, and deviating from them would seem to require some effort. In such cases, even rather weak notions of rational justification carry some weight. Such settings include, in particular, situations where computer protocols are announced and appropriate software that acts "as expected" is available. From the onset, we should note that our notions are intended for cases where the existing standard notions of games with incomplete information do not apply: ex-post Nash equilibria do not exist, and no reasonable common prior can be assumed (i.e. we seek "worst-case" notions as in computer science rather than Bayesian notions common in economics).

5.1 Set-Nash Equilibria

We first describe the set equilibrium notions for games with complete information, and then explain how to extend them to a setting of incomplete information, which suits our needs here. There are n players, where each player i has a strategy space S_i. The outcome of the game is given by the n utility functions $u_i : S \to \Re$ where $u_i(s_i, s_{-i})$ denotes i's payoff he plays strategy s_i and the others play the strategy tuple s_{-i}. The basic assumption is that, given that the other players play s_{-i}, player i will choose a strategy $s_i \in \text{argmax}\{u_i(s_i, s_{-i})\}$.

In our setting, a set of recommended strategies, R_i, is defined for each. The motivating scenario is where it is known that if all players i play recommended strategies then the outcome is "good" in some sense. E.g., in our case, the obtained social welfare approximates the optimal one (therefore we do not put any emphasis on the minimality of the sets; see the discussion on related literature below for details). We would like to capture the notion that the sets R_i are in equilibrium. In other words, formalize when can it be said that given that other players $j \neq i$ all play strategies in R_j, then player i also rationally plays some strategy in R_i.

We give four definitions below, all maintain the spirit of this "set equilibrium" notion, in order of increasing strength. Some of these notions have been defined before in the literature in the context of complete information games – we discuss this below in section 5.1.1. All of the following definitions behave the same on the two extreme cases: When each R_i is a singleton set ($\forall i | R_i = 1$) then they are equivalent to Nash equilibrium. When R_i is the entire strategy space ($R_i = S_i$) then they are trivially satisfied.

Definition 11

1. *We say that R_i are in "Set-Nash equilibria" (in the pure sense) if for every i, every $s_{-i} \in R_{-i}$, and every $s_i \in S_i$ there exists $r_i \in R_i$ such that $u_i(r_i, s_{-i}) \geq u_i(s_i, s_{-i})$. I.e. for very tuple of recommended strategies there exists a best response strategy in the recommended set.*

2. *We say that R_i are in "Set-Nash equilibria" (in the mixed sense) if for every i, for every series of distributions π_j on R_j for all $j \neq i$, and every $s_i \in S_i$ there exists $r_i \in R_i$ such that $u_i(r_i, s_{-i}) \geq E_{\{\pi_j\}_{j \neq i}}[u_i(s_i, s_{-i})]$. I.e for every series of distributions on the recommended strategies of the other players there exists a best response in the recommended set. This definition captures an expected-utility scenario,* over all possible priors.

3. *We say that $\{R_i(\cdot)\}$ are in "Set-Nash equilibria" (in the mixed-correlated sense) if for every i, for every π on $s_{-i} \in R_{-i}$, and every $s_i \in S_i$, there exists $r_i \in R_i$ such that $u_i(r_i, s_{-i}) \geq E_\pi[u_i(s_i, s_{-i})]$. This definition extends the previous one in the sense of allowing the other players to correlate strategies.*

4. *We say that R_i are in "Set-Domination equilibria" if for every i, and every $s_i \in S_i$ there exists $r_i \in R_i$ such that for every $s_{-i} \in R_{-i}$, we have that $u_i(r_i, s_{-i}) \geq u_i(s_i, s_{-i})$. I.e. for every unrecommended strategy, there is a recommended strategy that that is not worse-off, as long as others act as recommended.*

These definitions extend to games with incomplete information in a straightforward way. Each player i has a privately known type (input) $t_i \in T_i$. No probability distribution is assumed on $T = T_1 \times ... \times T_n$. The utility functions now depend on the player's type, as well ($u_i : T_i \times S \to \Re$, where $u_i(t_i, s_i, s_{-i})$ denotes i's payoff when his type is t_i, he plays strategy s_i and the others play the strategy tuple s_{-i}). The set of recommended strategies may now depend on the player's type, i.e. $R_i : T_i \to 2^{S_i}$, and we denote also $R_i(*) = \cup_{t_i \in T_i} R_i(t_i)$. All four definitions are modified so that the condition specified should now hold for all possible types t_i. In addition, the best response r_i must exist in player i's recommended set *according to his true type*, $R_i(t_i)$, and this r_i should be a best response to any tuple of strategies out of $R_{-i}(*)$ (i.e. the requirement holds for all possible type realizations of the other players). For example, the first definition is altered so that the set functions $R_i(\cdot)$ are in "Set-Nash equilibria" (in the pure sense) if for every i, every t_i, every $s_{-i} \in R_{-i}(*)$, and every $s_i \in S_i$ there exists $r_i \in R_i(t_i)$ such that $u_i(t_i, r_i, s_{-i}) \geq u_i(t_i, s_i, s_{-i})$.

In all definitions, we require the existence of a pure recommended strategy $r_i \in R_i(t_i)$. One can in principle relax the definition to allow r_i to be a mixed strategy, i.e. a probability distribution on $R_i(t_i)$. It is easy to verify that this does not change the first three definitions (the best mixed strategy is always a pure one), while for the Set-Domination definition, this will weaken it to become equivalent to Set-Nash for correlated strategies (using von-Neuman's max-min principle, in the sense of Yao showing equivalence between distributional complexity and probabilistic one).

The first three definitions suffer from the same caveats of regular Nash-equilibria, in particular noting that inequalities are not strict. Thus for example one can have any of these equilibria in strictly dominated strategies. More refined notions may require that strategies in $R_i(t_i)$ are undominated, or even that all undominated best-responses are in $R_i(t_i)$.

Another refinement is to show that the best response is in $R_i(t_i)$ even when other players' strategies reside in a wider class than $R_{-i}(*)$ (this may be interesting also when i assumes only partial rationality of the other players). One may formally define the wider set of acceptable strategies $A_i \subseteq S_i$, where $R_i(*) \subseteq A_i$, and replace the quantification of $s_{-i} \in R_{-i}(*)$ in the definition with $s_{-i} \in A_{-i}$.

In this work we use the basic definition (and drop the qualifier "in the pure sense" hereafter). In addition, all our Set-Nash strategies are undominated, and one can show that they are best response to a set of acceptable strategies wider than $R_{-i}(*)$. Indeed, we feel that an interesting problem we leave open is to find Set-Nash equilibria that contain best responses for mixed recommended strategies as well.

5.1.1 Related notions in the Game-Theory literature

The game theory literature defines and discusses similar notions to the above set equilibria notions for games with complete information. Most study is done on the existence and uniqueness of minimal such equilibria. We are not aware of any study in the setting of implementation theory, that examines such notions with respect to the quality of the outcome they yield.

Shapley [14] defines a notion of "a saddle" for two-person zero-sum games, which is almost the same as the Set-Domination notion (but the inequalities are strict). Shapley shows that there always exists a unique minimal saddle in a zero-sum game (the strictness of the inequalities are crucial for this), but does not address the quality of the obtained outcome. Duggan and Le Breton [6, 8] define a "mixed saddle", which allows mixed strategies in the definition. As we noted above, this is actually equivalent to the definition of Set-Nash in the correlated sense. Their results are again for the complete information case (mainly for zero-sum games, and for voting procedures). Duggan and Le Breton [7] develop a general approach to construct "choice sets". They require both an "outer stability", which resembles our logic of constructing a set equilibria, and also require an "inner stability", in order to have a minimal choice set. We replace this inner stability with a requirement on the quality of the outcome. This of-course can be done in our context of implementation theory, but not in their context of normal form games with complete information. Basu and Weibull [3] study sets of strategies that contain all their best replies (a "curb" set). Voorneveld [15] defines a "prep-set", which is equivalent to our definition of Set-Nash in the mixed sense. He studies the existence of a minimal such set in games with complete information. Bernheim [4] considers "point rationalizability", in which only pure strategies are considered. Although the motivation behind rationalizable strategies is different than the motivation of set equilibria, it is interesting to parallel the shift from rationalizable strategies (in which mixed strategies are allowed) to point rationalizabilty, to the shift from Set-Nash in the mixed sense to Set-Nash in the pure sense, that we make.

5.2 Implementation in Set-Nash equilibria

As our context is the framework of implementation theory, we wish to formally specify how the notion of Set-Nash equilibria fits in, in parallel to classical results. We do this for the basic definition of Set-Nash, but the entire discussion follows through for all four definitions in an immediate way. The setting contains a set of outcomes/alternatives, A, from which we have to choose one outcome. The choice depends on the players types $t \in T$, according to some social choice correspondence $F : T \rightarrow 2^A$. In our example, A is the set of all valid allocations of items to players, and $F(t)$ outputs all allocations that are 3-approximations w.r.t t. This social correspondence represents the fact that our goal is to obtain a 3-approximation of the welfare, and any allocation that obtains this will satisfy us. All the classic definitions from implementation theory can be adapted to our Set-Nash definition:

Definition 12 *Given $F : T \rightarrow 2^A$, an implementation in Set-Nash equilibrium is a mechanism with strategy sets $S_1, ..., S_n$, and an outcome function $g(s_1, ..., s_n) \in A$, such that there exists a Set-Nash equilibrium $\{R_i(\cdot)\}_i$ that satisfies that $g(s) \in F(t)$ for* **all** *$s \in R(t)$.*

Notice that we cannot hope to require that *all* equilibria will produce results according to F, as there always exists the trivial set-equilibria that contains all strategies.

The celebrated revelation principle states that whenever we can implement a social function in some equilibrium, we can also implement it using a direct revelation implementation, in which the strategy space of the players is simply to reveal their type. For our "set equilibrium" notion, we can have an "extended direct revelation" implementation which is "extended truthful":

Definition 13 *An implementation is an "extended direct revelation implementation" if the strategies of the players are of the form (t_i, l_i), where $t_i \in T_i$, and l_i represents any additional information.*

An extended direct revelation implementation is "extended truthful" (in Set-Nash equilibrium) if there exists a Set-Nash equilibrium in which $R_i(t_i) = (t_i, *)$, i.e. the player declares his true type in every one of his recommended strategies.

Proposition 1 (An extended revelation principle) *Every function $F : T \rightarrow 2^A$ that can be implemented in Set-Nash equilibrium can be implemented by an extended truthful implementation.*

proof: Given an implementation M to F in Set-Nash equilibrium, we build an extended truthful implementation M', that encapsulates M, as follows. Let $R_i(t_i)$ be the recommended strategies of M. Then the strategy space of a player in M' is to specify his type t_i, and a strategy in $R_i(t_i)$. The mechanism then uses M' with the specified strategies to determine the result. It is immediate to verify that the sets $R'_i(t_i) = \{(t_i, s_i) \mid s_i \in R_i(t_i)\ \}$ are indeed a Set-Nash that fits the definition. ∎

It is worth pointing out that our auctions, which are not direct revelation, have an interesting extended direct revelation counterpart – we describe this in section 6.1 below.

5.3 Ignorable Extensions of Games

This section formalizes a concept used in our proof of main theorem, below. In the proof, we first describe an extended truthful mechanism that implements a 3-approximation, and then show that each of our ascending auctions has "inside" it a semi-myopic mechanism. In this section, we describe this type of a building block more generally.

When one actually attempts to implement a game as a software protocol, it often turns out that the set of strategies that is available to players has grown: the protocol that allows a player to play any strategy in S_i turns out to enable also other strategies, informally ones that are "locally" like a valid strategy $s_i \in S_i$, but that do not correspond to any single valid strategy. These new strategies may open up new strategic behaviors. We will specify the requirements from such implementations needed to maintain Set-Nash equilibria.

Formally, given a game with incomplete information $G = (T, S, u)$ (where T, S, u are the players' type space, the players' strategies, and the players' utility functions, as described in section 5.1 above) we say that $\bar{G} = (T, \bar{S}, \bar{u})$ is an extension of G if $S_i \subseteq \bar{S}_i$ for all i and $\bar{u}_i(t_i, s) = u_i(t_i, s)$ for all $t_i \in T_i$ and $s \in S$ (i.e. $\bar{u}$ when restricted to S is identical to u).

Clearly a strategy that was best response in G need not be a best response in $\bar{G}$ since the new strategies $\bar{S}_i \setminus S_i$ may be better. "Ignorable" extensions of G will not allow such better strategies:

Definition 14 *We say that $\bar{G}$ is an* **ignorable extension** *if for all i, all $t_i \in T_i$, all $s_{-i} \in S_{-i}$ and all $\bar{s}_i \in \bar{S}_i$ there exists $s_i \in S_i$ such that $u_i(t_i, s_i, s_{-i}) \geq u_i(t_i, \bar{s}_i, s_{-i})$. I.e. if others play an original strategy then I have an original strategy which is a best response.*

Proposition 2 *If $\{R_i(\cdot)\}$ are a Set-Nash equilibrium of G and $\bar{G}$ is an ignorable extension of G then $\{R_i(\cdot)\}$ are a Set-Nash equilibrium of $\bar{G}$.*

We point out that, although these notions where related to the notion of Set-Nash equilibria, we can, in an immediate and similar way, define ignorable extensions to any one of the other three definitions of equilibria.

6 A Strategic Analysis of our Auctions

6.1 Semi-Myopic Mechanisms

We now devise an extended direct revelation auction with our two basic building blocks: it has a Set-Nash equilibrium, and, for these equilibrium strategies, the auction is a semi-myopic allocation rule. We will later use this to show that both our online ascending auctions also have such a structure.

Definition 15 *We define the* **semi-myopic mechanism** *as follows:*

Strategy space: *Each player declares, as he arrives, his value, his deadline, and a tentative deadline between his arrival time and his deadline. The variable $d(i,t)$ holds i's tentative deadline if t is not larger than his tentative deadline, otherwise $d(i,t)$ equals his final deadline.*

Winner determination at time t: *Let A_t, S_t, and f_t be the natural parallels of the notions in definition 9, where the deadline of each player in A_t is $d(i,t)$. The mechanism allocates item t to some player in f_t (this choice may depend on the contents and structure of A_t, S_t, and f_t).*

Prices: *For each player i, the mechanism maintains a tentative price for each time t, $p_t(i)$, as follows: If $i \notin S_t$ then $p_t(i) = 0$. For any $i \in S_t$, let*

$$c_t(i) = max\{v(j) \mid j \in A_t \setminus S_t,\ S_t \setminus i \cup j \text{ is independent w.r.t items } t, ..., M\}. \tag{2}$$

For any $i \in f_t$, the mechanism sets $p_t(i) = c_t(i))$. For any $i \in S_t \setminus f_t$, the mechanism may set any price $p_t(i) \in [0, c_t(i)]$. The winner i of time t pays $max_{r(i) \le t' \le t} p_{t'}(i)$.

The recommended strategies: *In a recommended strategy, i declares his true value and deadline at time $r(i)$, and may declare any tentative deadline.*

Lemma 5 *When all players play recommended strategies according to their true types then the semi-myopic mechanism is a semi-myopic allocation rule.*

proof: Fix any time t. We need to show that the mechanism chooses a player with value at least v_t^*. Let f_t^{true} be the "true" one, i.e. the relevant set computed with the true player deadlines, and let S_t, f_t be the actual sets computed by the mechanism according to the declared tentative deadlines. If $f_t^{true} \subseteq S_t$ then, by the prefix construction process described in section A.1, and since tentative deadlines are not larger than true ones, $f_t \subseteq f_t^{true}$, and the claim follows. Otherwise there is some $j \in f_t^{true} \setminus S_t$, and so for every $i \in f_t$, $v(i) \ge v(j) \ge v_t^*$ (recall that the declared values are the true ones), as claimed. ■

Theorem 4 *The semi-myopic mechanism Set-Nash implements a 3-approximation of the welfare.*

proof: We prove the theorem by the following claims:

Lemma 6 *For any player i, and any $s_{-i} \in R_{-i}(*)$, i has a best response to s_{-i} in $R_i(t_i)$.*

proof: Let σ be the scenario in which all players besides i play s_{-i}, and i does not show up at all. Let

$$t^* = argmin_{r_i \le t \le d_i}\{v_t^*(\sigma)\}. \tag{3}$$

Notice that player i can win and pay exactly $v_{t^*}^*$ by arriving at time t^*, declaring any value larger than $v_{t^*}^*$, and a deadline equals to t^*.

Claim 6 *t^* and $v^*_{t^*}$ does not depend on the choice of the winner $i \in f_t$ of time $t \in [r(i), d(i)]$ (where the winners prior to time $r(i)$ are fixed).*

proof: By contradiction, assume that there exist two different scenarios, σ_1, σ_2, that differ only in the choice of the winners (notice that the f_t's themselves might become different during the scenario run due to a previous choice of different winners). Let $v^*(\sigma_i) = min_{r(i) \leq t \leq d(i)}\{v^*_t(\sigma_i)\}$, and let t^*_i be the minimal time in which $v^*(\sigma_i)$ is obtained.

We first assume w.l.o.g. that $v^*_{t^*_2}(\sigma_1) > v^*_{t^*_2}(\sigma_2) = v^*(\sigma_2)$. Let us justify this. If $v^*(\sigma_1) \neq v^*(\sigma_2)$ then w.l.o.g. $v^*(\sigma_1) > v^*(\sigma_2)$ and therefore also $v^*_{t^*_2}(\sigma_1) > v^*(\sigma_2)$. If $v^*(\sigma_1) = v^*(\sigma_2)$ then, by the contradiction assumption, $t^*_1 \neq t^*_2$, so w.l.o.g. $t^*_2 < t^*_1$. Therefore $v^*_{t^*_2}(\sigma_1) > v^*(\sigma_1) = v^*(\sigma_2)$, as needed. Notice also that from this it follows that $t^*_2 > r(i)$, as $A_{r(i)}(\sigma_1) = A_{r(i)}(\sigma_2)$.

Since $v^*_{t^*_2}(\sigma_1) > v^*_{t^*_2}(\sigma_2)$ then $f_{t^*_2}(\sigma_1) \neq f_{t^*_2}(\sigma_2)$, and therefore, by the prefix properties of section A.1, $f_{t^*_2}(\sigma_1) \not\subseteq S_{t^*_2}(\sigma_2)$. Fix some $j \in f_{t^*_2}(\sigma_1) \setminus S_{t^*_2}(\sigma_2)$. Since $v(j) > v^*_{t^*_2}(\sigma_1)$ it follows that $j \notin A_{t^*_2}(\sigma_2)$. This implies that, in σ_2, j is the winner of some time $t' < t^*_2$, i.e. $j \in f_{t'}(\sigma_2) = P_{t'}(t', \sigma_2)$. As $d(j) \geq t^*_2$ then $P_{t'}(t', \sigma_2) = P_{t'}(t^*_2, \sigma_2)$. By claim 28 of section A.1, it therefore follows that $v^*_{t'}(\sigma_2) \leq v^*_{t^*_2}(\sigma_2)$, contradicting the choice of t^*_2. ∎

Claim 7 *i's price in any strategy s_i is at least $v^*_{t^*}$ (where the other players play s_{-i}).*

proof: Recall that σ denotes the scenario in which i does not show up at all. Let σ' be the scenario in which i plays some strategy s_i and the others play s_{-i}. Denote by t_0 the minimal t with $i \in f_t(\sigma')$. We claim that there exists a scenario σ'', that differs from σ only in the choice of winners in f_t, such that $A_{t_0}(\sigma') = A_{t_0}(\sigma'') \cup i$. This follows by the an inductive argument: At time $t < t_0$, $A_t(\sigma') = A_t(\sigma'') \cup i$. Since $i \notin f_t(\sigma')$ then, by claim 30, $f_t(\sigma') = f_t(\sigma'')$. Choose the winner in σ'' to be the winner of σ'. Therefore $A_{t+1}(\sigma') = A_{t+1}(\sigma'') \cup i$, and the inductive claim follows.

Now, at time t_0, since $i \in f_{t_0}(\sigma')$ then, by claim 30, there exists some $j \in A_{t_0}(\sigma') \setminus S_{t_0}(\sigma')$ such that $j \in f_{t_0}(\sigma)$. Therefore i's price is at least $v(j) \geq v^*_{t_0}(\sigma'') \geq v^*_{t^*}(\sigma)$ (where the last inequality follows by claim 6, and the lemma follows. ∎

Claim 8 *The (recommended) strategy of arriving at time $r(i)$, declaring the true value and deadline and declaring a tentative deadline equals to t^* is a best response of i against s_{-i}.*

proof: If $v(i) \leq v^*_{t^*}$ then i cannot possibly gain positive utility, as claim 7 shows, and indeed any recommended strategy will not allocate any item to i.

If $v(i) > v^*_{t^*}$ then, if player i arrives at time t^* and declares tentative deadline t^* he will win item t^* for a price of $v^*_{t^*}$. Let σ be the scenario in which i does not show up at all and σ' be the scenario in which i arrives at $r(i)$ and declares tentative deadline t^*. We claim by induction that, for any $t < t^*$, the winners of σ and σ' are identical, and that i's tentative price is at most $v^*_{t^*}$. Therefore i will win item t^* for a price of $v^*_{t^*}$, and the claim follows. For any $t < t^*$, we have by claim 28 and the construction of t^* that $min_{j \in P_t(t^*, \sigma)}\{v(j)\} \leq min_{j \in P_{t^*}(t^*, \sigma)}\{v(j)\} = v^*_{t^*}(\sigma) < v^*_t(\sigma)$. By the maximality of $S_t(\sigma')$ it follows that, in σ', i replaces the minimal player in $P_t(t^*, \sigma)$, therefore $f_t(\sigma) \subseteq S_t(\sigma')$, and so $f_t(\sigma) = f_t(\sigma')$. By claim 6 we can assume w.l.o.g. that the winner has not changed in the transition from σ to σ'. i's price at time t is (at most, as the mechanism has some freedom in setting this) $min_{j \in P_t(t^*, \sigma)}\{v(j)\} \leq v^*_{t^*}$, and therefore i's final price was not affected as well. ∎

6.2 Bad Examples

We would like to show, by an example, that the recommended strategies of the semi-myopic mechanism do not contain best responses to mixed strategies. We will only show it for correlated mixed strategies, i.e. it does not contain a best response against a distribution over all $R_{-i}(*)$. We will start with a basic problematic scenario, and then add to it a second scenario, together obtaining the counter example.

The basic problematic scenario demonstrates that a player might be tempted to arrive later, or to declare a deadline higher than his true one, although this is not his best response:

Example 2 *Consider the following scenario, where (v, d) denotes a player with value v and deadline d):*

- *At time 1 arrive players $(\epsilon, 1), (x_1, 4), (x_2, 4), (x_3, 4), (x_4, 4)$.*
- *At time 2 arrive players $(y_1, 2), (y_2, 3)$.*
- *At time 3 arrive players $(z_1, 5), (z_2, 5)$.*
- *At time 4 arrives a (very large) player $(z_3, 4)$.*

where the values satisfy: $\epsilon < x_2, x_3 < y_1 < x_1 < z_1, z_2 < y_2 < x_4 < z_3$.

If all players declare their true value and tentative deadline equals to their true deadline, a semi-myopic mechanism can choose the winners (first to last) x_1, y_1, y_2, z_3, z_1. So player x_4 looses. However, if he delays his arrival to time 2, or, equivalently, declares a deadline of 5, the winners will be $\epsilon, y_2, x_4, z_1, z_3$, so x_4 will win, with price x_1. Notice, however, that this is not his best response. His best response, to arrive at time 1 and declare tentative deadline 1, is still of-course recommended.

Example 3 *Let scenario 1 be the scenario of example 2, where we consider the decisions faced by x_4, and scenario 2 be as follows:*

- *At time 1 arrive player $(x, 1)$ and our player $(x_4, 4)$.*
- *At time 2 arrive player $(x, 2)$.*
- *At time 3 arrive player $(x, 3)$.*

(where $x = x_4 - \epsilon$). The best response of x_4 to scenario 1 is to arrive at time 1 and declare deadline 1. The best response to scenario 2 is to arrive at time 1 and declare a deadline of 4 (thus winning item 0 with price 0). Now suppose that player x_4 knows/estimates that both scenarios have probability half. Then, a quick calculation shows that if x_4 plays some recommended strategy (and thus arrives at time 1) with tentative deadline lower than 4, then with probability half (for scenario 2) he will win of the items 1 to 3 with a resulting utility (i.e. value minus price) of ϵ. If his tentative deadline will be 4 then with probability half (for scenario 1) he will lose. Therefore, any recommended strategy has resulting utility at most $(x_4 + \epsilon)/2$. However, if x_4 will arrive at time 2 and will declare deadline 4, a non-recommended strategy, his resulting utility will be half times $x_4 - 0$ (for scenario 2) plus half times $x_4 - x_1$, better than $(x_4 + \epsilon)/2$ for small enough ϵ.

6.3 The Online Iterative Auction has a Set-Nash Equilibrium

We now show that our Online Iterative Auction is an ignorable extension of a semi-myopic mechanism, thus having a Set-Nash equilibrium which approximates the welfare, according to theorem 4. For this, we need to refine our intuitive definition:

Definition 16 (The Online Iterative Auction) *We apply the following modifications to Def. 2:*

Prices: *The auction maintains a tentative price $p_t(i)$ for each player i at time t, as follows: if i is a tentative winner at the end of the iterations of time t then $p_t(i)$ equals to the tentative price of i's item, otherwise $p_t(i) = 0$. The winner i of time t pays $max_{r(i) \leq t' \leq t}\{p_{t'}(i)\}$.*

Recommended strategies: *i's strategy is recommended if i chooses a tentative deadline $d \leq d(i)$, plays myopically (as in Def 3) with value $v(i)$ and deadline d in all times $r(i) \leq t \leq d$, and plays myopically with value $v(i)$ and deadline $d(i)$ in all times $t > d$.*

It is not hard to verify that these recommended strategies are semi-myopic.

Theorem 5 *The Online Iterative Auction is an ignorable extension of a semi-myopic mechanism.*

Corollary 2 *The Online Iterative Auction Set-Nash implements a 3-approximation of the welfare.*

Proof (of theorem): We first prove that the iterative auction is an extension of a semi-myopic mechanism. We will then show that this extension is ignorable.

Lemma 7 *If all players i play strategies in $R_i(*)$ then the iterative auction is a semi-myopic mechanism.*

proof: We need to map every recommended strategy of the iterative auction to a strategy of the semi-myopic mechanism, such that the result of the iterative auction (winners plus payments) will match the criteria of a semi-myopic mechanism. This is done as follows. At time t, map every players that plays myopically with (v, d) to a type (v, d), and denote this set of types as A_t. Let S_t be the optimal allocation of items $t, ..., M$ to the players of A_t. All we need to show is that the iterative auction selects a winner from f_t and sets correct payments. In what follows, we use the notion of a prefix and the claims of section A.1. Let $Y[t, ..., M]$ and $p_t[t, ..., M]$ be the tentative allocation and prices of the iterative auction with the myopic strategies, at the end of time t. For any $d \geq t$, let $P_Y(d)$ be the appropriate prefix of Y, according to definition 20. Define $l(d) = min\{d' \geq t \mid P_Y(d') = P_Y(d)\}$, and

$$c_t(d) = max\{v(j) \mid j \in A_t \setminus Y \text{ and } d(j) \geq l(d)\ \}.$$

(notice that, by abuse of notation, we have defined both $c_t(d)$ for an item $d \in \{t, ..., M\}$, and $c_t(i)$ for a player i. Those are two differently defined terms, although we will see below that they are equal, for $d = Y[i]$).

Claim 9 $p_t(d) \geq c_t(d)$.

proof: Fix any $j \in A_t \setminus Y$ with $d(j) \geq l(d)$. If $d(j) \geq d$ then j has positive value for receiving d. Since j is myopic, it therefore follows that $p_t(d) \geq v(j)$. If $d(j) < d$, then since $P_Y(l(d)) = P_Y(d)$, By the construction of $P_Y(l(d))$, since it is equal to $P_Y(d)$, then there exist players $i_1, ..., i_k$ and items $t_1, ..., t_k$ such that, for any index $x \in \{1, ..., k\}$, $i_x = Y[t_x]$, $d(i_x) \geq t_{x+1}$, $t_1 \leq l(d)$, and $t_k = d$. Since $d(i_x) \geq t_{x+1}$ it follows that $p_t(t_x) \leq p_t(t_{x+1})$, otherwise i_x would have placed his name on item t_{x+1}. Therefore $p_t(d) = p_t(t_k) \geq p_t(t_1)$. Since $t_1 \leq l(d) \leq d(j)$ it follows that $p_t(t_1) \geq v(j)$, and the claim follows. ∎

Claim 10 *If $p_t(d) > p_{t-1}(d)$ then $p_t(d) \leq c_t(d)$.*

proof: Suppose by contradiction that d is the maximal one with $p_t(d) > c_t(d) + \epsilon$, for some small $\epsilon > 0$. Thus, at some point in the iterative process of time t, the price of item d was $c_t(d) + \epsilon/2$, and then some player, j, placed his name on item d, further increasing its price. Let $X[t, ..., M]$ be the tentative allocation at this point, just before j's action. Let us examine the identity of this player j. Notice that any item $d' < d$ has price at most $c_t(d)$, as any player that placed his name on d could have placed his name on d'. We first claim that $Y[l(d), ..., d] \subseteq X[l(d), ..., d]$. Otherwise, fix some $i \in X[l(d), ..., d] \setminus Y[l(d), ..., d]$. If $i \in Y[d+1, M]$ then i placed his name on an item with price strictly larger than $c_t(d) \geq c_t(d+1) \geq p_t(d+1)$ which is larger or equal to the current price of item $d+1$, a contradiction to the myopic behavior of i. If $i \in Y[t, l(d)-1]$ then, by the prefix properties, $d(i) < l(d)$, a contradiction. And if $i \in A_t \setminus Y$ with $d(i) \geq l(d)$ then $v(i) \leq c_t(d)$ by definition, therefore i placed his name on an item with price higher than his value, again a contradiction. Therefore $Y[l(d), ..., d] \subseteq X[l(d), ..., d]$. Now, j places his name on item d. But $j \notin Y[l(d), ..., d]$ as these players are already tentative winners, and $j \notin A_t \setminus Y[l(d), ..., d]$, by repeating exactly the same arguments from above, thus reaching a contradiction. ∎

Claim 11 *$Y = S_t$, and, for any $d \geq t$ and $i = Y[d]$, $c_t(d) = c_t(i)$ (as defined in eq. 2).*

proof: We first show that, for any $j \in A_t \setminus Y$, $Y \setminus i \cup j$ is independent w.r.t items $t, ..., M$ if and only if $d(j) \geq l(d)$. Since $Y[t, ..., l(d) - 1]$ is a prefix, any allocation X that contains it cannot allocate an item $\leq l(d) - 1$ to player $j \notin Y[t, ..., l(d) - 1]$. Therefore $d(j) \leq d$. In the other direction, if $d(j) \leq d$ then we can simply allocate d to player j instead of to i, thus having an allocation to $Y \setminus i \cup j$. Otherwise, $l(d) \leq d(j) \leq d$, and we can use the exact same chain argument of claim 9 to obtain an allocation, when replacing i with j.

From this and claim 9 we have that, for any $i \in Y$ and $j \in A_t \setminus Y$ such that $Y \setminus i \cup j$ is independent w.r.t items $t, ..., M$, $v(i) \geq p_t(d) \geq c_t(d) \geq v(j)$. This property immediately implies, by the matroid basic properties, that Y is the optimal allocation. By using the above claim again we now get that $c_t(d) = c_t(i)$. ∎

From this last claim it follows that the winner of time t belongs to f_t, as $f_t \subseteq S_t = Y$, and therefore all first $|f_t|$ items of S_t must be sold to the players of f_t. It remains to show that the prices charged by the auction match the criteria of the semi-myopic mechanism.

Claim 12 *In the Online Iterative Auction, the winner i of time t pays $max_{r(i) \leq t' \leq t}\{c_{t'}(i)\}$.*

Let $p_t(i)$ be i's tentative price at time t. Let t' be such that $i = Y[t']$. By the above claims, $p_t(i) = p_t(t') \geq c_t(t') = c_t(i)$. We additionally show that either $p_t(i) = p_{t-1}(i)$ or $p_t(i) = c_t(i)$, and the claim will follow. Assume $p_t(i) \neq p_{t-1}(i)$. Therefore i must have placed his name on item t'

during the iterative process of time t. Thus $p_t(t') > p_{t-1}(t')$, and, by the above claims, it follows that $p_t(i) = p_t(t') = c_t(t') = c_t(i)$. ∎

This concludes the proof of Lemma 7. ∎

We now continue with the proof of the theorem. By Lemma 7 it follows that the set $R(*)$ of the Online Iterative Auction forms a semi-myopic mechanism, and so the Online Iterative Auction is an extension of the semi-myopic mechanism. Fix any player i and and combination of recommended strategies of the other players, $s_{-i} \in R_{-i}(*)$. We need to show that i has best response to s_{-i} in $R_i(*)$. Since all players beside i are myopic with tentative deadline and then with final deadline, we can map them to types (v, d) as in Lemma 7. Let σ be this scenario, where i does not show up at all, and define t^* as in equation 3 of the proof of lemma 6 of section 6.1. Now suppose i plays some strategy $\bar{s}_i$, and denote this scenario by σ'. Let $Y_t(\sigma), Y_t(\sigma')$ be the tentative winners at time t in scenarios σ, σ', respectively. Let t_0 be the first time t such that $f_t(\sigma) \not\subseteq Y_t(\sigma')$. Therefore, for every $t < t_0$, σ' chooses a winner from $f_t(\sigma)$, and by claim 6 we can assume w.l.o.g. that σ and σ' choose the same winner. Therefore $A_{t_0}(\sigma') = A_{t_0}(\sigma) \cup i$. Now suppose that $i = Y_{t_0}(\sigma')[d]$ for some d (if $i \notin Y_{t_0}(\sigma')$ then $d = M + 1$).

Claim 13 *Then $Y_{t_0}(\sigma')[t_0, ..., d-1]$ is independent with respect to items $t_0 + 1, ..., d$.*

proof: By contradiction, let $f \subseteq Y_{t_0}(\sigma')[t_0, ..., d-1]$ be its minimal prefix. Fix any $j \in f_{t_0}(\sigma) \setminus Y_{t_0}(\sigma')$. Since $j \in A_{t_0} \setminus Y_{t_0}(\sigma')$ we have that the tentative price of item t_0 in σ' is at least $v(j)$. By a chain argument as in claim 10 if follows that every $j' \in f$ has value at least than $v(j)$. Since $j \in f_{t_0}(\sigma)$ it then follows that $j' \in Y_{t_0}(\sigma)$. Thus $f \subseteq Y_{t_0}(\sigma)$. Therefore $f_{t_0}(\sigma) = f \subseteq Y_{t_0}(\sigma')$, a contradiction. ∎

From this, and by using again the chain argument of claim 10 we get that d's price is at least $v^*_{t_0}(\sigma) \geq v^*_{t^*}(\sigma)$. As i can win and pay $v^*_{t^*}(\sigma)$ by a strategy in $R_i(*)$ (e.g. arriving at time t^* and bidding only on item t^*), the claim follows. ∎

This concludes the proof of the Theorem. ∎

6.4 The Sequential Japanese Auction has a Set-Nash Equilibrium

To show that our Sequential Japanese Auction is an ignorable extension of a semi-myopic mechanism, we need to modify payments similarly to the modification of the Online Iterative Auction. For this, we need to handle simultaneous "drop" announcements more carefully: At any price level p, several players may want to drop. Furthermore, this may be an on-going process, as after one player drops, another one now wants to drop as well. We need to determine more accurately the order among them. This information is used in order to determine f_t (interestingly, we are not able to compute S_t entirely, only f_t, which is enough).

Definition 17 (The Sequential Japanese Auction) *The basic auction structure remains the same as in Def 6. Two additional points should be handled:*

Simultaneous "drop" announcements: *Define $D(p, n)$ as the set of players (among those who did not drop yet), that wish to drop when the price level is p and the number of remaining players is n. At every price level p, the auction solicits drop announcements by repeatedly accepting only one drop announcement out of $D(p, n)$, and decreasing n by 1.*[14] *When $D(p, n) = \emptyset$, the price increases. The winner is, as before, the last remaining player.*

[14] E.g. if $D(p, n) = X$ then some $i \in X$ is chosen to be dropped, $X \setminus i \subseteq D(p, n-1)$ and $i \notin D(p, n-1)$.

Prices: *Prices $p_t(i)$ for every player i at every time t are maintained as follows: Let k be the number of non-drop-outs just before the price ended its time-t ascend, at a level of p^*. Let $D(p^*, k), D(p^*, k-1), ..., D(p^*, 1)$ be the order of drop-outs at this level. Define the critical number $x^* = \min\{0 < x < k \ : \ |D(p^*, x+1)| = 1\}$, and $D^* = \cup_{x \le x^*} D(p^*, x)$. For any player i, if $i \in D^*$ set $p_t(i) = p^*$, otherwise $p_t(i) = 0$. The winner i of time t pays $max_{r(i) \le t' \le t}\{p_{t'}(i)\}$.*

Recommended strategies: *i's strategy is recommended if he arrives at $r(i)$, choose a tentative deadline $d \le d(i)$, plays myopically with parameters $v(i), d$ until time d, and plays myopically with parameters $v(i), d(i)$ thereafter.*

Again, these recommended strategies are semi-myopic.

Theorem 6 *The Sequential Japanese Auction is an ignorable extension of a semi-myopic mechanism.*

Corollary 3 *The Sequential Japanese Auction Set-Nash implements a 3-approximation of the welfare.*

Proof (of theorem): The theorem will follow by the following claims.

Lemma 8 *If all players i play strategies in $R_i(*)$ then the Sequential Japanese Auction forms a semi-myopic mechanism.*

proof: Let p^* be the last price reached by the auction of time t, and suppose there are k players that did not drop out just before p^* was reached.

Claim 14 *Fix any $j \in A_t \setminus S_t$. As long as j does not drop, then every $i \in P_t(d(j))$ does not drop.*

proof: By contradiction, let $i \in P_t(d(j))$ be the first to drop, say at price p. Since j did not drop, $v(j) \ge p$. By the maximality of S_t, $v(i) > v(j)$. Thus i did not drop because of the price. But the number of non-dropped players is at least $|P_t(d(j))| + 1 > d(i)$. Therefore i could not have dropped at this point, a contradiction. ∎

Claim 15 $p^* = max\{\ v(j) \mid j \in A_t \setminus S_t\ \}$.

proof: Let j^* be the player with maximal value among those in $A_t \setminus S_t$. By the previous claim, j^* will drop because the price will reach his value, as $|P_t(d(j^*))| \ge d(j^*)$. Thus $p^* \ge v(j^*)$. Suppose by contradiction that $p > v(j^*)$, and choose some p in between. Thus, when the price reaches p, all the non-drop-outs belong to S_t. Consider the one that receives, according to S_t, the latest item. The number of non-drop-outs is smaller than his deadline, so he will drop. The one that receives the item before last will next drop, by the same argument, and so on. Therefore the price will not increase beyond p, a contradiction. ∎

Claim 16 *For any $i \in f_t$, $p^* = c_t(i)$.*

proof: For any $j \in A_t \setminus f_t$, $S_t \setminus i \cup j$ is independent: choose an allocation in which i receives item t, and then remove i and allocate t to j. Therefore the claim follows from the previous claim, and from the definition of $c_t(i)$. ∎

Claim 17 *For any l', $|D(p^*, l') \cup \cdots \cup D(p^*, 1)| = l'$.*

proof: Since $D(p^*, l'), ..., D(p^*, 1)$ includes only players that did not actually drop before phase (p^*, l'), and there are exactly l' of those, then $l' \geq |D(p^*, l') \cup \cdots \cup D(p^*, 1)|$. On the other hand, every player among the l' players that did not drop yet will drop in some phase $D(p^*, l'), ..., D(p^*, 1)$, so $l' \geq |D(p^*, l') \cup \cdots \cup D(p^*, 1)|$. ∎

Claim 18 *If $|D(p^*, l' + 1)| = 1$ then $D(p^*, l') \cup \cdots \cup D(p^*, 1)$ is a prefix.*

proof: Since $|D(p^*, l' + 1)| = 1$ then any $j \in D(p^*, l') \cup \cdots \cup D(p^*, 1)$ has deadline $d(j) < t + (l' + 1) - 1$, i.e. $d(j) \leq t + l' - 1$. Since $|D(p^*, l') \cup \cdots \cup D(p^*, 1)| = l'$ it follows from claim 24 that $D(p^*, l') \cup \cdots \cup D(p^*, 1)$ is a prefix. ∎

Claim 19 *Let x^* be the critical number of drop-outs, and $D^* = \cup_{x \leq x^*} D(p^*, x)$, as in def. 17. Then $D^* = f_t$.*

proof: Let $l = |f_t|$. Notice that, for any $l' > l$, $f_t \cap D(p^*, l') = \emptyset$: If $i \in f_t$ then $v(i) > p^*$ and $d(i) \leq |f_t| + t - 1 < l' + t$, so i will not drop. This, in turn, implies that a player in f_t will drop in one the of phases $(p^*, l), ..., (p^*, 1)$, so $f_t \subseteq D(p^*, l) \cup \cdots \cup D(p^*, 1)$. Since $|D(p^*, l) \cup \cdots \cup D(p^*, 1)| = l$, we conclude that $f_t = D(p^*, l) \cup \cdots \cup D(p^*, 1)$. It is left to show that $x^* = l$. As $D(p^*, l) \subseteq f_t$ and $f_t \cap D(p^*, l+1) = \emptyset$ then $D(p^*, l+1) \cap D(p^*, l) = \emptyset$. This implies that $|D(p^*, l+1)| = 1$, so $x^* \leq l$. But if $x^* < l$ then $D(p^*, x^*) \cup \cdots \cup D(p^*, 1) \subsetneq f_t$ is a prefix, contradicting the minimality of f_t (by claims 25, 27). Therefore $x^* = l$ and $D^* = f_t$. ∎

From all the above, the proof of the claim immediately follows: First, the winner belongs to $D^* = f_t$. Second, all time t prices for players not in f_t equal 0, and for players in f_t, time t prices equal $p^* = c_t(i)$, i.e. as required by the price rules of the semi-myopic mechanism. ∎

We now continue with the proof of the theorem. Using the above claim, it only remains to show that, fixing some player i and some strategies $s_{-i} \in R_{-i}(*)$ of the other players, i has a best response in $R_i(*)$. Consider some strategy s_i of i. Let t_0 be the first time in which i enters D^*. We first notice that, in every time prior to t, i can wave participation without affecting the winner: If the price when i participates reached a level p^*, then clearly, when i does not participate the price cannot rise above p^*. By definition, a player in D^* will not drop before there will be at most $|D^*| - 1$ other non-drop-outs (as the price does not reach his value). Therefore the last non-drop-outs will be exactly all players in D^*, and so the winner will be the same.

Now suppose the price level at time t, in which i entered D^*, is p^*. Therefore i's price will be at least p^*. We claim that, by arriving at time t and playing the fixed confidence strategy $(p^*, 1)$, i can win item t for a price p^*. Since this strategy is in $R_i(*)$, the claim will follow. To see this, observe that $|D(p^*, x)| > 1$ for any $1 < x < x^*$ (since $D(p^*, x) \cap D(p^*, x - 1) \neq \emptyset$). Therefore, even if i will not be willing to drop out until being the last non drop out, all others will drop out at price p^*, and so i will win t and will pay p^*. ∎

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A Useful Properties of Offline Allocations and Matroids

This section summarizes useful properties that we have used throughout our proof. Most of the properties here are new, and are interesting in our context. For completeness, we begin with a short introductory summary of Matroids and their relevant properties.

Definition 18 (A Matroid) *A Matroid is a finite set S and a collection $I \subseteq 2^S$ of independent sets, such that:*

1. $\emptyset \in I$
2. *If $X \in I$ and $Y \subseteq X$ then $Y \in I$.*
3. *If $X, Y \in I$ and $|X| = |Y| + 1$ then there exists $j \in X \setminus Y$ such that $Y \cup j \in I$.*

If $X \subseteq S$ but $X \notin I$ then it is a dependent set. A base of a matroid is a maximal independent set, and a cycle is a minimal dependent set.

Claim 20 *The offline allocation of M items among a set A of players is a matroid, where S is the set of players, and a subset X of players is independent if there exists an allocation of (part of) the items to all the players in X.*

proof: The first two conditions of the matroid are trivially satisfied. Let us verify the third one. Let X, Y are be two independent sets with $|X| > |Y|$. We first claim that there exists allocations for X, Y such that, for any $j \in X \cap Y$, j receives the same item in both allocations. To see this, start from arbitrary two allocations, and choose some $j \in X \cap Y$. Suppose j receives items t_1, t_2 in the allocations $1, 2$, where $t_1 < t_2$. Suppose that player j' receives item t_2 in allocation 1. Then we can swap players j and j' in allocation 1, so that j will receive item t_2 (this is valid as we know he can receive this item) and j' will receive item t_1 (this is valid as $t_1 < t_2$). Notice that we have strictly decreased the number of players in $X \cap Y$ that receive different items, and so repeating this implies the result. Now, choose some item t which is being allocated for X but not allocated to any player of Y. Suppose that t is allocated to j in the allocation of X. By our assumption, $j \notin Y$, and so $Y \cup j$ is independent: use the previous allocation of Y, and allocate item t (that beforehand was not allocated) to j. ∎

The following claim lists some useful matroid properties. For extensive discussion and proofs, see e.g. the textbook [16].

Claim 21 *Let $M = (S, I)$ be a matroid. Then:*

1. *If $X, Y \in I$ and $|X| < |Y|$ then there exists $Z \subseteq X \setminus Y$ such that $|X \cup Z| = |Y|$ and $X \cup Z \in I$.*
2. *If B_1, B_2 are bases then $|B_1| = |B_2|$.*
3. *If B_1, B_2 are bases, then, for any $j \in B_1 \setminus B_2$ there exists $j' \in B_2 \setminus B_1$ such that $B_1 \setminus j \cup j' \in I$ and $B_2 \setminus j' \cup j \in I$.*

The following claims are slight alterations of classical properties:

Claim 22 *Let $X, Y \in I$, and $X \not\subseteq Y$. Then, for any $j \in Y \setminus X$ such that $X \cup j \notin I$ there exists $j' \in X \setminus Y$ such that $X \setminus j' \cup j \in I$ and $Y \setminus j \cup j' \in I$.*

proof: If $|X| = |Y|$ then we can assume w.l.o.g. that both are bases (as $I' = \{ Z \in I \mid |Z| \leq |X| \}$ are also the independent sets of a matroid), and the claim immediately follows.

If $|X| > |Y|$ then assume, as before, that X is a base. There exists $Z \subseteq X \setminus Y$ such that $B = Y \cup Z$ is a base. Since $j \in Y \setminus X$ then $j \in B \setminus X$ and so there exists $j' \in X \setminus B$ such that $X \setminus j' \cup j \in I$ and $B \setminus j \cup j' \in I$. Since $Y \subseteq B$ and $j' \in Y \setminus X$ as well, the claim follows.

If $|X| < |Y|$ then assume that Y is a base, take some $Z \subseteq Y \setminus X$ such that $B = X \cup Z$ is a base, and notice that $j \notin Z$ as $X \cup j \notin I$. Thus we can essentially repeat the above logic: j is also in $Y \setminus B$ so there exists $j' \in B \setminus Y$ such that $B \setminus j' \cup j \in I$ and $Y \setminus j \cup j' \in I$. Since $B \setminus Y = X \setminus Y$, and $X \subset B$, then the claim follows. ∎

Claim 23 *Let B be a base of the matroid, and $Y \in I$ such that $|B \setminus Y| = 1$. Then $|Y \setminus B| \leq 1$.*

proof: $|B| \geq |Y| = |B \cap Y| + |Y \setminus B| = |B| - |B \setminus Y| + |Y \setminus B| = |B| - 1 + |Y \setminus B|$. Therefore $|Y \setminus B| \leq 1$, as claimed. ∎

A.1 Some Useful Properties of Offline Allocations

For the following discussion, it will be convenient to assume the following ϵ-assumption: There exists many small valued players in A_t that desire any one of the items $t, ..., M$.

Definition 19 (A prefix) *A subset $X \subseteq S_t$ is called a prefix if it is a prefix of any allocation $S_t[1, M]$ of S_t.*

Claim 24 *$X \subseteq S_t$ is a prefix if and only if for all $j \in X$, $d(j) \leq t + |X| - 1$.*

proof: Suppose first that X is a prefix, and, by contradiction, that there exists some $j \in X$ with $d(j) > t + |X| - 1$. Let $S_t[t, M]$ be some allocation of S_t. Since $j \in X$ and X is a prefix then j is allocated some item $\leq t + |X| - 1$. Suppose player j' is allocated item $d(j)$. Then we can switch between j and j' and have an allocation in which X is not a prefix, a contradiction. In the other direction, if $X \subseteq S_t$ and $d(j) \leq t + |X| - 1$ for any $j \in X$ then, in any allocation, $j \in S_t[t, t + |X| - 1]$. Therefore $X \subseteq S_t[t, t + |X| - 1]$, and since $|S_t[t, t + |X| - 1]| = |X|$ then it follows that $S_t[t, t + |X| - 1] = X$, i.e. it is a prefix. ∎

Definition 20 *For any $t \leq d \leq M$, we build the set of players $P_t(d)$ using the following process (fix any allocation of S_t):*

1. *Let $x_0 = d$.*
2. *For $i > 0$, define inductively $x_i = max\{ d(j) \mid j \in S_t[t, x_{i-1}]\}$.*
3. *Let k be some index such that $x_{k+1} = x_k$, and fix $P_t(d) = S_t[t, x_k]$.*

Claim 25 *$P_t(d)$ is the prefix with minimal length among all prefixes with length $\geq d - t + 1$.*

proof: First notice that, from the ϵ-assumption it immediately follows that $|P_t(d)| = x_k - t + 1$. Also notice that, by our construction, any $j \in P_t(d)$ has $d(j) \leq x_k = t + |P_t(d)| - 1$. Therefore, by claim 24, $P_t(d)$ is a prefix. Suppose by contradiction that there exists a prefix P' with $d \leq t + |P'| - 1 < x_k$. Choose index i such that $x_i \leq t + |P'| - 1 < x_{i+1}$. But then, by the construction process of $P_t(d)$, we must have a player in P' with deadline at least x_{i+1}, contradicting claim 24. ∎

Claim 26 *$j \in P_t(d)$ if and only if there exists an allocation of S_t in which $j \in S_t[t, d]$.*

proof: If $j \in S_t[t, d]$ then by definition $j \in P_t(d)$. Let us verify the other direction. Fix any allocation of S_t, and compute $P_t(d)$ by that allocation. Assume $j = S_t[d']$ for some $d' > d$ (otherwise the claim immediately follows). Let j_i be the player that determined x_i. Then we have $j_1 \in S_t[t, d]$. Consider the following allocation replacements: allocate item x_1 to player j_1 (this is his deadline, so this is valid), j_2 will get item x_2, ... , j_k will get item x_k. Finally, allocate j's item to j_{k+1} (that received x_k), and allocate j_1's item to j. Therefore we have an allocation in which j receives some item $\leq d$, as claimed. ∎

Claim 27 *$f_t = P_t(t) = P_t(|first_t| + t - 1)$.*

proof: If $j \in first_t$ then there exists an allocation of S_t such that $j = S_t[t]$. Since $P_t(t)$ is a prefix of $S_t[1, M]$ then $j \in P_t(t)$. On the other hand, claim 26 tells us that for any $j \in P_t(t)$ there exists an allocation such that $j = S_t[t]$, and therefore $j \in f_t$. We conclude that $f_t = P_t(t)$. From claim 25 we now get also that $P_t(t) = P_t(|first_t| + t - 1)$, as $P_t(t)$ is a prefix with length $|first_t|$. ∎

Claim 28 *For any t, d with $t < d$, $min_{j \in P_{t+1}(d)}\{v(j)\} \geq min_{j \in P_t(d)}\{v(j)\}$.*

proof: We will actually show that $min_{j \in P_{t+1}(d)}\{v(j)\} \geq min_{j \in P_t(d) \setminus ON[t]}\{v(j)\}$. Let $x = |P_t(d)| + t - 1$, the last item allocated to a player in $P_t(d)$. By the above claims, for any $j \in P_t(d)$, $d_j \leq x$, and $x \geq d$. Let j be the player with minimal value in $P_{t+1}(d)$, and assume by contradiction that $v(j) < min_{j \in P_t(d) \setminus ON[t]}\{v(j)\}$. Therefore $j \notin P_t(d) \setminus ON[t]$. Consider some allocation of S_{t+1} such that j receives item $\leq d$. Now consider $P_t(d) \setminus ON[t]$ and $S_{t+1}[t + 1, x]$. These are two bases of the matroid over items $t + 1, ..., x$. Since $j \in S_{t+1}[t + 1, x] \setminus (P_t(d) \setminus ON[t])$ then there exists $j' \in P_t(d) \setminus ON[t] \setminus S_{t+1}[t+1, x]$ such that $S_{t+1}[t+1, x] \setminus j \cup j'$ is independent (w.r.t items $t+1, ..., x$). As $d_{j'} \leq x$, it follows that $j' \notin S_{t+1}$, and therefore $S_{t+1} \setminus j \cup j'$ is independent as well. As $j' \in A_{t+1}$, and by the maximality of S_{t+1}, we must have $v(j) > v(j') \geq min_{j \in P_t(d)}\{v(j)\}$, a contradiction. ∎

Claim 29 *f_t is independent w.r.t items $t + 1, ..., M$ if and only if S_t is independent w.r.t items $t + 1, ..., M$.*

proof: Since $f_t \subseteq S_t$ then the right to left direction is immediate. Let us verify the other direction, i.e. that if f_t is independent w.r.t items $t + 1, ..., M$ then so is S_t. Let $\tilde{A}_t, \tilde{f}_t, \tilde{S}_t$ be the variables after adding many ϵ players, as in the ϵ-assumption. By the optimality of S_t it follows that $S_t \subseteq \tilde{S}_t$ (when ϵ is small enough). As $\tilde{f}_t$ is a prefix, it cannot be independent w.r.t items $t + 1, ..., M$. Thus there exists $j \in \tilde{f}_t \setminus f_t$. By definition, $\tilde{S}_t \setminus j$ is independent w.r.t items $t+1, ..., M$, and therefore $S_t \setminus j$ is independent w.r.t items $t + 1, ..., M$. If $j \in S_t$ this will therefore imply $j \in f_t$, a contradiction. Thus j is an ϵ player, and $S_t \subseteq \tilde{S}_t \setminus j$. Since $\tilde{S}_t \setminus j$ is independent w.r.t items $t + 1, ..., M$, then this implies that so is S_t, as needed. ∎

Claim 30 *Let $A'_t = A_t \cup j'$. Let S_t, S'_t and f_t, f'_t be derived from A_t, A'_t, respectively. Then:*

1. *If $j' \in f'_t$ then $f_t \setminus S'_t \neq \emptyset$.*
2. *$f_t \neq f'_t$ if and only if $j' \in f'_t$.*

proof: From the prefix properties it immediately follows that, if $f_t \subseteq S'_t$ then $f'_t = f_t$, and thus the first claim follows. This also implies the right to left direction of the second claim. We are left to show that, if $f_t \neq f'_t$ then $j' \in f'_t$. By the maximality of S_t, S'_t it follows that either $S_t = S'_t$, or $S'_t = S_t \setminus j \cup j'$ for some $j \in S_t \setminus S'_t$. Since $f_t \neq f'_t$, the latter alternative must hold. If $j \notin f_t$ then $f_t \subseteq S'_t$, implying that $f'_t = f_t$, a contradiction. Thus $j \in f_t$. Therefore there exists an allocation with $j = S_t[t]$. Since $S'_t = S_t \setminus j \cup j'$ then there exists an allocation with $j' = S'_t[t]$ (simply use the previous allocation, changing only the player who receives item t from j to j'). By definition, this implies that $j' \in f'_t$. ∎